PROJECT SQUID

TECHNICAL REPORT ARAP-7-P-1

THE FOUNDATIONS OF NONEQUILIBRIUM STATISTICAL MECHANICS

Vol. 1 of 2 Vols.

By

Guido Sandri

Aeronautical Research Associates of Princeton, Inc.

DEPARTMENT OF AEROSPACE ENGINEERING
SCHOOL OF ENGINEERING AND APPLIED SCIENCE
UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VIRGINIA

Project SQUID is a cooperative program of basic research relating to Jet Propulsion. It is sponsored by the Office of Naval Research and is administered by the University of Virginia through Contract Nonr 3623(00), NR-098-038.

June 1963

Technical Report ARAP-7-P-1

PROJECT SQUID

A COOPERATIVE PROGRAM OF FUNDAMENTAL RESEARCH
AS RELATED TO JET PROPULSION
OFFICE OF NAVAL RESEARCH, DEPARTMENT OF THE NAVY

Contract Nonr 3623(00), NR-098-038

THE FOUNDATIONS OF NONEQUILIBRIUM STATISTICAL MECHANICS*

Vol. 1 of 2 Vols.

by
Guido Sandri
Aeronautical Research Associates of Princeton, Inc.
Princeton, New Jersey

June 1963

PROJECT SQUID HEADQUARTERS
DEPARTMENT OF AEROSPACE ENGINEERING
SCHOOL OF ENGINEERING AND APPLIED SCIENCE
UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VIRGINIA

*To be published in Annals of Physics

TABLE OF CONTENTS

Part I

Part II

ABSTRACT

In this paper a novel philosophy of irreversible dynamics is formulated. This philosophy stems from the fact that the kinetic equations available (Boltzmann, Landau, Bogolubov-Balescu-Lenard) are essentially exact and cannot be improved. That is, for kinetic gases (those whose behavior is characterized by that of one typical particle) these equations constitute closed, statistically complete knowledge.

This thesis is demonstrated by using a technique that separates completely the different time components exhibited by the evolution of a gas when an appropriate parameter (characteristic of the regime in which the gas is found) is small. The expansion in this parameter is pushed up to its breaking point marked by the presence of an intrinsic divergence. With our technique we can pinch off the series at this point and remain with a closed, finite system of equations. The argument is made more compelling by the fact that the same divergence occurs for all gaseous regimes (short-range, weak-coupling, dilute weak-coupling and Debye).

We furthermore prove that not all isolated gases will eventually become kinetic. In fact, the necessary and sufficient conditions that the initial departure from equilibrium must satisfy for kineticity to eventually set in ("absence of parallel motion") are deduced directly from the Liouville equation.

When statistical information about a gas is needed beyond that afforded by the knowledge of the motion of the average particle a new asymptotic expansion of the Liouville equation must be devised. Thus, for example, the preponderance of three-body collisions demands knowledge of the evolution of the average pair of particles. For this situation a pair-kinetic expansion is introduced. A hierarchy of increasingly more informative descriptions of a gas is thus envisaged.

As a by-product of our analysis, (i) we have found the limits

of validity of Bogolubov's assumption of synchronization and of his boundary condition and (ii) we have proved the equivalence of Kirkwood's time averaging procedure (made systematic) with Bogolubov's technique.

ACKNOWLEDGMENTS

I wish to express my gratitude to several colleagues. First of all, W. Hayes, A. Grossmann, J. McCune, T. Morse, and R. Sullivan, who have made invaluable contributions to my thinking. It is also a pleasure to acknowledge enlightening discussions with C. Donaldson, F. Dyson, M. Goldberger, M. Rosenbluth, R. Stora, A. Wightman, and G. E. Uhlenbeck, and especially a vigorous and stimulating exchange of ideas with E. Frieman during the course of this research.

I would also like to thank Profs. E. Boldt and P. R. Weiss for their useful comments and for their hospitality during a series of lectures that I gave at Rutgers University in the Spring term 1961-1962, and to Messrs. B. Maxfield, T. Simon, S. Skalski, and R. Turoff for putting at my disposal their lecture notes.

This work was sponsored partly by Project Squid which is supported by the Office of Naval Research, Department of the Navy, under contract Nonr-3623(S-7), Phase 1, and partly by the Advanced Research Projects Agency under Air Force Office of Scientific Research of the Office of Aerospace Research Contract No. AF 49(638)-1140. Reproduction in full or in part is permitted for any use of the United States Government.

SECTION I

INTRODUCTION

The aim of this paper is to obtain from first principles the equations that govern the evolution in time of a system that contains a large number of particles. That is, we are concerned with systems such as a macroscopic portion of a body (not necessarily a gas) or a large stellar system like a galactic cluster.

The description is of course statistical. We want to be able to describe the evolution of arbitrary initial departures from thermodynamical equilibrium.

The behavior of our systems over very long periods of time is of particular interest. We believe that, in virtue of Gibbs' general H theorem, most isolated systems will tend to thermodynamic equilibrium. We are therefore interested in obtaining solutions of the fundamental equations of motion which are valid for times comparable to that needed for the system to reach equilibrium.

The irreversibility in the evolution of our systems will of necessity play a very prominent role.

The equation that we have to solve in order to describe our "many-body" system is the Schrödinger equation for N bodies:

$$i \frac{\partial \Psi^N}{\partial t} = H^N \Psi^N \qquad (1.1)$$

or, more generally, when a statistical ensemble is considered, the equation for the density matrix:

$$i \frac{\partial \rho^N}{\partial t} = \left[H^N, \rho^N \right] \qquad (1.2)$$

where H^N is the Hamiltonian operator and

$$[H^N, \rho^N] \equiv H^N \rho^N - \rho^N H^N$$

The complete quantum mechanical description is not yet of full interest because our understanding of truly non-equilibrium quantum effects (like non-equilibrium phonon-electron interactions in solids) is as yet rudimentary.

We shall confine ourselves therefore to classical (i.e. non-quantum) phenomena here. The fundamental equation for our study is accordingly the Liouville equation:

$$\frac{\partial F^N}{\partial t} + \mathcal{H}^N F^N = 0 \tag{1.3}$$

where F^N is the probability density in the phase space of N bodies and $\mathcal{H}^N$ is the Poisson bracket operator.

All three of the equations written down are of the form:

$$\frac{\partial F}{\partial t} + \mathcal{H} F = 0 \tag{1.4}$$

where $\mathcal{H}$ is a linear operator acting on F. This fact makes the problem of finding solutions valid for long times very similar in classical and in quantum systems. The class of functions is very different in the two cases: a Hilbert space in quantum theory of equation (1.1) and a limit of a sequence of (real) Banach spaces in classical theory.

In Section 2 we show that a direct perturbation expansion of equations of the type (1.4) leads in general to approximations which are not uniformly valid for all times.

In Section 3 a new mathematical frame work is set up for handling problems in which a small parameter plays a prominent role. The relevant formulae are summarized in Section 4.

After a brief outline of the statistical mechanics that we propose as correct (Section 5) we start our discussion of the Liouville equation (Section 6). The concepts of kinetic theory and the basic regimes for which kinetic equations are available are also defined in Section 6.

Then we take up successively the weak coupling (Sections 7 and 8), short range (Sections 9, 10, and 11) and Debye expansions (Section 12).

Generalized master equations as well as the kinetic theories are deduced directly (i.e. without use of the BBGKY hierarchy) from the Liouville equation in Section 13.

Pair-kinetic equations are introduced in Section 14.

In Section 15 some aspects of inhomogeneous gases and of the transition into fluid flow are discussed.

Finally in Section 16 the equivalence of Kirkwood's and Bogolubov's techniques is proved.

SECTION II

NON-UNIFORMITY IN PERTURBATION THEORY

We rewrite (1.4) by means of a decomposition

$$H = H^0 + \varepsilon H^1 \tag{2.1}$$

where ε is a small number:

$$\varepsilon << 1 \tag{2.2}$$

and both H^0 and H^1 are linear. We now try to find solutions to

$$\frac{\partial F}{\partial t} + H^0 F = -\varepsilon H^1 F \tag{2.3}$$

by assuming F analytic in ε :

$$F = F^0 + \varepsilon F^1 + \varepsilon^2 F^2 + \cdots \tag{2.4}$$

Furthermore, the initial value of F(t), F(0), is given and, for simplicity, does <u>not</u> contain ε

$$\frac{\partial F(0)}{\partial \varepsilon} = 0 \tag{2.5}$$

From (2.3) we obtain

$$\frac{\partial F^0}{\partial t} + H^0 F^0 = 0 \tag{2.6}$$

$$\frac{\partial F^1}{\partial t} + H^0 F^1 = -H^1 F^0 \tag{2.7}$$

$$\frac{\partial F^2}{\partial t} + H^0 F^2 = -H^1 F^1 \tag{2.8}$$

and so on. The general solution of (2.6) is, using (2.5)

$$F^0(t) = e^{-\mathcal{H}^0 t} F(0) \tag{2.9}$$

Substituting (2.9) into (2.7) and integrating:

$$F^1(t) = -e^{-\mathcal{H}^0 t} \int_0^t \left[e^{+\mathcal{H}^0 \tau} \mathcal{H}^1 e^{-\mathcal{H}^0 \tau} \right] d\tau F(0) \tag{2.10}$$

Similarly (2.8) yields

$$F^2(t) = +e^{-\mathcal{H}^0 t} \int_0^t \left[e^{+\mathcal{H}^0 t} \mathcal{H}^1 e^{-\mathcal{H}^0 \tau} \right] \cdot$$

$$\cdot \int_0^\tau \left[e^{+\mathcal{H}^0 \tau'} \mathcal{H}^1 e^{-\mathcal{H}^0 \tau'} \right] d\tau' d\tau F(0) \tag{2.11}$$

and so on.

These formulae correspond to the expansion in powers of the operator $\exp(-\mathcal{H}t)$

$$e^{-(\mathcal{H}^0 + \varepsilon \mathcal{H}^1)t} = e^{-\mathcal{H}^0 t} - \varepsilon e^{-\mathcal{H}^0 t} \int_0^t \left[e^{+\mathcal{H}^0 \tau} \mathcal{H}^1 e^{-\mathcal{H}^0 \tau} \right] d\tau +$$

$$+ \varepsilon^2 e^{-\mathcal{H}^0 t} \int_0^t \left[e^{+\mathcal{H}^0 \tau} \mathcal{H}^1 e^{-\mathcal{H}^0 \tau} \right] \int_0^\tau \left[e^{+\mathcal{H}^0 \tau'} \mathcal{H}^1 e^{-\mathcal{H}^0 \tau'} \right] d\tau' d\tau + \ldots \tag{2.12}$$

For the very complex problems of interest to us here the series (2.4) is very rarely convergent. We demand that it should always by asymptotic. We must therefore insure that F^0 is the "main" contribution to the answer and that εF^1 is a small correction to F^0, that is

$$\frac{\varepsilon F^1}{F^0} = O(\varepsilon) \quad \text{all } t \tag{2.13}$$

The condition (2.13) must hold for almost all values of the parameters of the system; for our statistical mechanical theory, for long times and over most of the phase space in particular. If we want an even better approximation than the one afforded by

$$F \approx F^0 + \varepsilon F^1 \tag{2.14}$$

we have to consider F^2 as well and demand that it lead to a small correction. For example, if

$$\frac{\varepsilon F^2}{F^1} = O(\varepsilon) \quad \text{all } t \tag{2.15}$$

the approximation

$$F \approx F^0 + \varepsilon F^1 + \varepsilon^2 F^2 \tag{2.16}$$

will be superior to (2.14). It is possible for (2.16) to be a good approximation even if (2.13) does not hold, but rather:

$$\frac{\varepsilon^2 F^2}{F^0 + \varepsilon F^1} = O(\varepsilon^2) \quad \text{all } t \tag{2.17}$$

By glancing at the explicit form of the functions F^0, F^1, and F^2 given by (2.9), (2.10), and (2.11) it will be clear that the conditions for a good approximation very often break down.

For example, if $\mathcal{H}^0$ and $\mathcal{H}'$ are simply numbers

$$F^1 = t e^{-\mathcal{H}^0 t} \mathcal{H}^1 F(0) \tag{2.19}$$

and

$$F^0 = e^{-\mathcal{H} t} F(0) \tag{2.18}$$

so that

$$\frac{\epsilon F^1}{F^0} = (\epsilon t) \mathcal{H}^1 \tag{2.20}$$

It is clear that this ratio is not of order ϵ after a time

$$t \gtrsim \frac{1}{\epsilon} \tag{2.21}$$

The simple perturbation expansion (2.4) is therefore inadequate.

A moment's thought will convince the reader that (1.3), the Liouville equation, is satisfied by the positions and momenta of the N bodies as well as by F^N. In this case (1.3) simply runs along the orbits backwards in time. The perturbation expansion leads then to what astronomers call "secular terms", that is orbital elements that increase with time. To remove this difficulty, Poincare devised a refinement of the perturbation expansion which has found important extensions outside of celestial mechanics (Poincare - Lighthill method (1)). The technique consists basically in expanding both the dependent and the independent variables in power series of the small parameter.

On the other hand, Bogolubov has refined an expansion technique (synchronization) invented by Enskog (2) to solve Boltzmann's equation, which is capable of giving approximations to (1.3) valid for times comparable with the time required for the approach to

equilibrium. The Enskog-Bogolubov technique basically consists in expanding the time derivative of the dependent variables in a power series in ϵ. Bogolubov's technique is valid only for the lowest significant approximation to the lowest moment of $\partial F^N/\partial t$ (i.e. $\partial F^1/\partial t$, to be discussed at great length later) and attempts at extending his calculations (3) have so far met with lack of success.

Bogolubov's synchronization assumption corresponds to selecting a certain class of physical systems (kinetic gases) described by the Liouville equation. The meaning and limitations of this choice will be made clear in the sequel, since in this paper we find the precise conditions for kineticity.

Instrumental for the understanding of these conditions is the analysis of perturbation theory for large times. The asymptotic analysis of perturbation theory, originated by Bogolubov, has been made more explicit by M. Rosenbluth and N. Rostoker (11) and recently by E. Frieman (5). The latter development, aside from deriving kinetic equations, illuminates their physical significance and allows for making explicit the dynamical approach to kineticity.

It is a most remarkable fact that the situations which can be handled by expanding the independent variable á la Poincare are intractable by expanding the time derivative á la Enskog and vice versa.

The purpose of the next section is to present an entirely new approach (method of extension) to the problem of obtaining significant (i.e. uniformly valid) approximations to functions which, like φ^N or F^N, depend on some small parameter. The approach is extremely general because it includes, among others, as very special and mutually incompatible cases both Poincare's and Enskog's techniques (cf. Fig. 1).

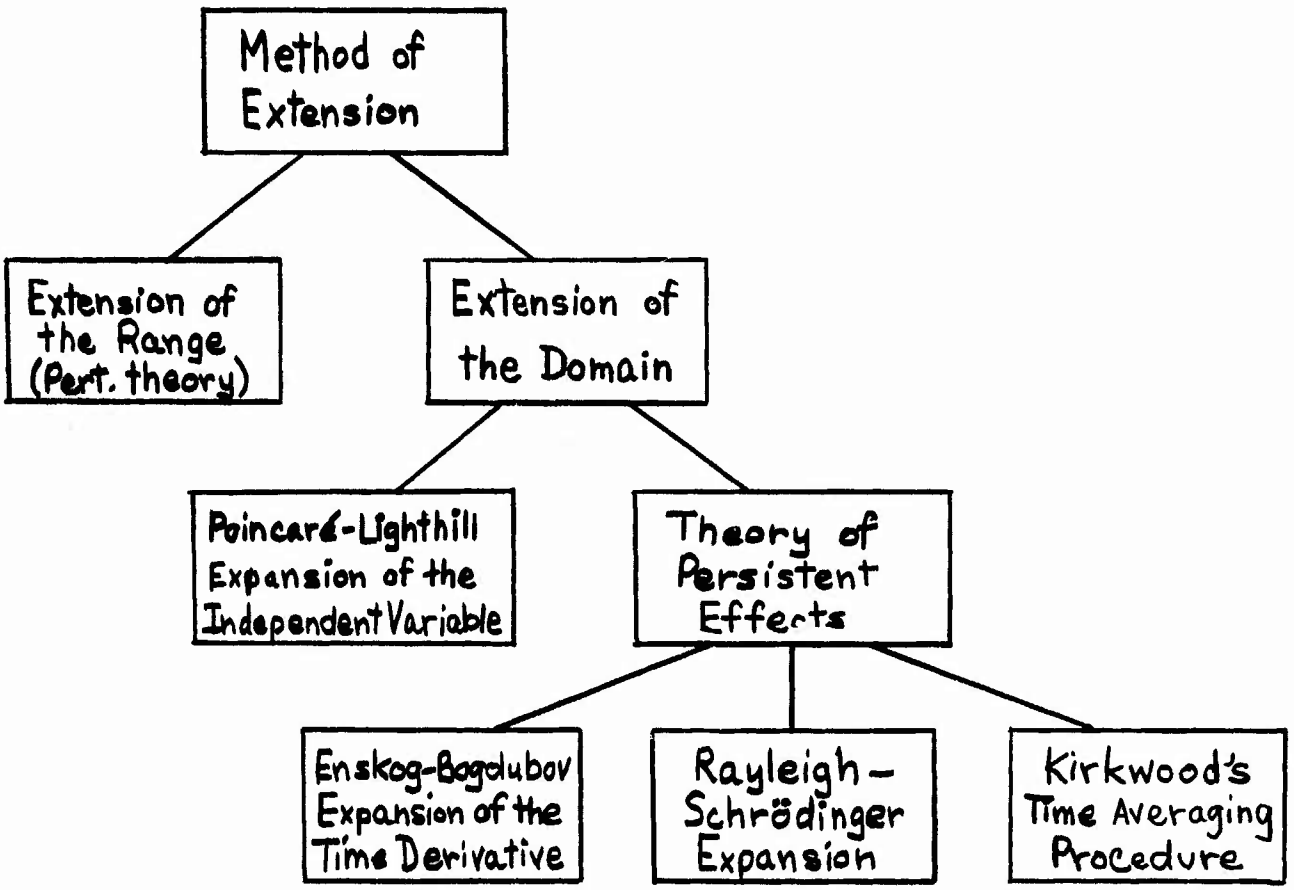

Fig. 1.

It will be clear that the general scheme is a framework into which one must inject information about the asymptotic behavior of the functions discussed. For the application carried out here, it will be seen that the direct perturbation expansion itself yields very important clues as to the relevant behavior of the functions of interest.

It is worth noting that the Enskog-Bogolubov expansion can be applied to the calculation of quantum mechanical stationary states. The result coincides with the Rayleigh-Schrödinger expansion (see Fig. 1). Successively more "persistent" effects correspond to making the solution stationary on successively slower clocks.

SECTION III

THE METHOD OF EXTENSION

This section is divided into two main parts: first the concept of extension is defined, then the framework for the applications is set up. In the next section a summary of formulae is given for the convenience of the reader.

A. The Concept of Extension

We consider a class of single valued mappings $\{\phi\}$ with domain δ and range ρ. The nature of the mappings, of the domain and of the range need not be specified now (they are of course assumed to be given). They will become clear to the reader shortly (Section 3C). Thus,

$$\phi : \quad \delta \longrightarrow \rho \tag{3.1}$$

We now consider any mappings E_δ of δ onto $I_\delta \subset \underline{\delta}$

$$E_\delta : \quad \delta \longrightarrow I_\delta \tag{3.2}$$

which is one to one and has an inverse E_δ^{-1}. The set $\underline{\delta}$ in which I_δ is embedded

$$\underline{\delta} \supset I_\delta \tag{3.3}$$

is presumed of high dimension. Thus,

$$\dim \underline{\delta} \geq \dim I_\delta = \dim \delta \tag{3.4}$$

For applications, dim $\underline{\delta}$ is often ∞. $\underline{\delta}$ is called the extension of the domain. We have thus injected δ into $\underline{\delta}$

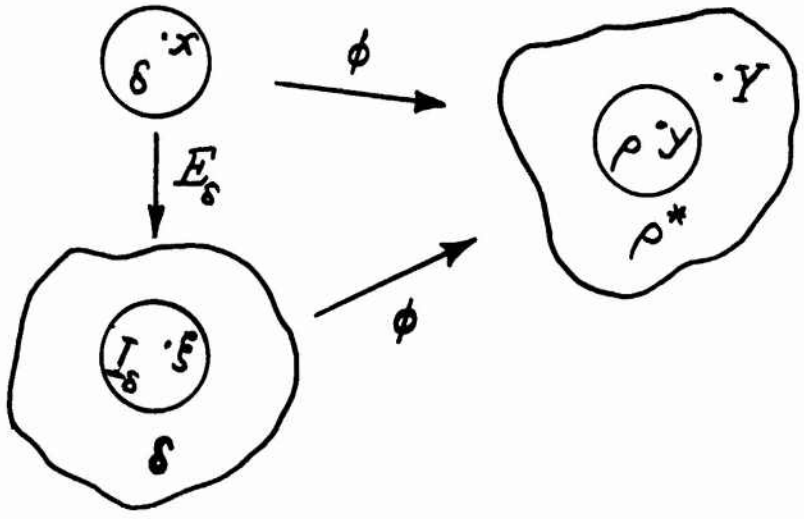

Fig. 2.

We now define an extension of ϕ , $\underline{\phi}$ as any single-valued mapping of $\underline{\delta}$ into ρ^* with

$$\rho^* \supset \rho \tag{3.5}$$

if and only

$$\underline{\phi} \circ E_\delta = \phi \tag{3.6}$$

where $\underline{\phi} \circ E_\delta$ means the composition of the two mappings E_δ and $\underline{\phi}$. We shall use the notation $\phi \Rightarrow \underline{\phi}$ (see fig. 2).

The analytic continuation of a complex valued function of a complex variable is an extension with an additional structure. In particular, a continuation like $\frac{1}{x} \Rightarrow \frac{1}{z}$ of a real valued function is an extension. The reader is urged to consider this familiar situation in the light of our definitions.

Consider further the identity mapping of δ onto itself

$$\mathcal{J}_\delta : \quad \delta \longrightarrow \delta \tag{3.7}$$

and the sets δ^* that include δ. $\underline{\mathcal{J}_\delta}$ is any extension of $\mathcal{J}_\delta$ if it satisfies (3.6). See fig. 3.

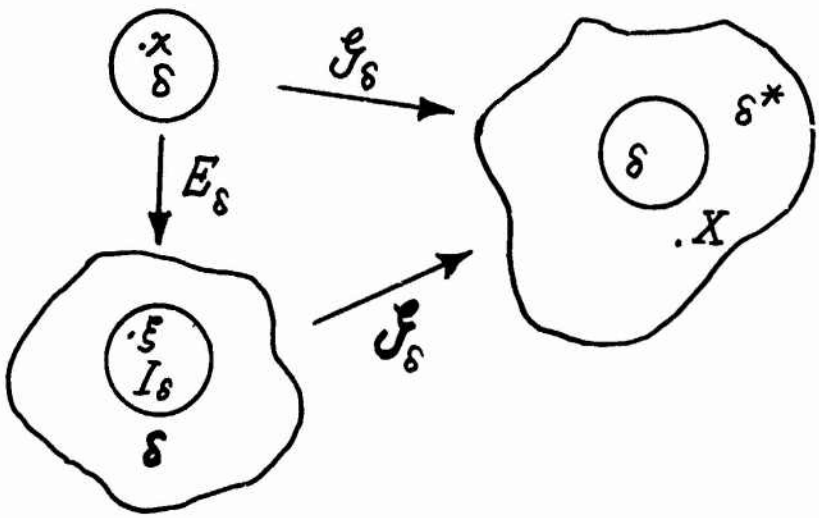

Fig. 3.

We then have, for the restriction of $\underline{\mathcal{J}_\delta}$ to I_δ, $[\underline{\mathcal{J}_\delta}]_{I_\delta}$

$$[\underline{\mathcal{J}_\delta}]_{I_\delta} : \quad I_\delta \longrightarrow \delta \tag{3.8}$$

Therefore

$$[\underline{\mathcal{J}_\delta}]_{I_\delta} = E_\delta^{-1} \tag{3.9}$$

One readily obtains the induction formula:

$$\phi \circ \left[\underline{\mathcal{J}_\delta}\right]_{I_\delta} = \left[\underline{\phi}\right]_{I_\delta} \tag{3.10}$$

This simple theorem is fundamental. It expresses the fact that all extensions of ϕ are "induced" by extensions of the identity mapping. Thus, it will be possible to understand the link between the Poincare-Lighthill expansion of the independent variable and the Enskog-Bogolubov expansion of the time derivative.

Let $x \in \delta$, $y \in \rho$, $\xi \in \underline{\xi}$, $Y \in \rho^*$, and $X \in \delta^*$ then (3.1), (3.2), the definition of $\underline{\phi}$, and (3.8) read

$$\phi: \quad x \longrightarrow y(x) \tag{3.11}$$

$$E_\delta: \quad x \longrightarrow \xi(x) \tag{3.12}$$

$$\underline{\phi}: \quad \xi \longrightarrow Y(\xi) \tag{3.13}$$

$$\underline{\mathcal{J}_\delta}: \quad \xi \longrightarrow X(\xi) \tag{3.14}$$

We attach to the point ξ local coordinates $\xi_i(\lambda)$ where λ is a function of x with local inverse (x and λ correspond in general to several dimensions). Thus, the <u>embedding equations</u> for the domain are

$$\xi_i = \xi_i(\lambda) \quad \text{with} \quad \lambda = \lambda(x) \tag{3.15}$$

The parameter λ allows for an arbitrary reparametrization of the domain. Such reparametrization may be equivalently performed before or after the embedding. It is worth noting that it is

essential to Lighthill's procedure that $\lambda(x) \neq x$.

In terms of our detailed notation (3.10) reads

$$y(X) = Y(\xi) \qquad \xi \in I_\delta , X \in \delta \tag{3.16}$$

The reason for introducing (3.11) to (3.16) is that the local coordinates ξ_i and the functions $X(\xi_i)$ are essential for explicit calculations.

The accurate representation of an extension is given by the commutative sequence

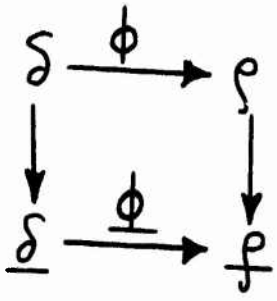

Fig. 4.

The simpler version given above is designed to present a complication unnecessary for our applications.

B. Important Special Cases

Two particularly useful examples of the embedding equations (3.15) are the following linear embeddings for one-dimensional real domains.

(i) "Singular" embedding. Let

$$\xi_0 \equiv x,\ \xi_1 = \alpha x, \ldots, \xi_n = \alpha^n x, \ldots \tag{3.17}$$

We have

$$\sum_{i=0}^{\infty} |\xi_i|^2 = \frac{x^2}{1-\alpha^2} \tag{3.18}$$

and therefore ξ is in a Hilbert space if $\alpha < 1$.

(ii) "Regular" embedding. Let

$$\xi_0 = x,\ \xi_1 = \frac{x}{\alpha}, \ldots, \xi_n = \frac{x}{\alpha^n}, \ldots \tag{3.19}$$

We have

$$\sum_{i=0}^{\infty} |\xi_i|^2 = x^2 \frac{\alpha^2}{\alpha^2 - 1} \tag{3.20}$$

and therefore ξ is in a Hilbert space if $\alpha > 1$.

The reason for the terminology will become clear in the sequel. We give now some simple examples of extended functions.

(i) Consider the mapping ϕ defined by

$$y(x) = e^{-\alpha x} \tag{3.21}$$

where x and y are real numbers, α real, < 1 . By means of (3.17) we can extend J_δ and hence ϕ in infinitely many different ways. For example,

(ia) with

$$\mathfrak{X}(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = \xi_0 \tag{3.22}$$

then

$$e^{-\alpha x} \Rightarrow e^{-\alpha \xi_0} \tag{3.23}$$

(ib) with

$$X(\xi_0, \xi_1, \ldots \xi_n, \ldots) = \xi_1/\alpha \tag{3.24}$$

then

$$e^{-\alpha x} \Rightarrow e^{-\xi_1} \tag{3.25}$$

(ic) with

$$X(\xi_0, \xi_1, \ldots, \xi_n, \ldots) \equiv \tfrac{1}{2}(\xi_0 + \xi_1/\alpha) \tag{3.26}$$

then

$$e^{-\alpha x} \Rightarrow e^{-\frac{1}{2}(\xi_0 + \xi_1/\alpha)} \tag{3.27}$$

(ii) Consider the mapping ϕ defined by

$$y(x) = e^{-x/\alpha} \tag{3.28}$$

where x, y are real as before and α is real and < 1.

With the help of (3.19) we can produce again infinitely many different extensions. For example

(iia) with

$$X(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = \xi_0 \tag{3.29}$$

then

$$e^{-x/\alpha} \Rightarrow e^{-\xi_0/\alpha} \tag{3.30}$$

(iib) with

$$\mathcal{X}(\xi_0, \xi_1, \ldots \xi_n, \ldots) = \alpha \xi_1 \tag{3.31}$$

then

$$e^{-x/\alpha} \Rightarrow e^{-\xi_1} \tag{3.32}$$

(iic) with

$$\mathcal{X}(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = \tfrac{1}{2}(\xi_0 + \alpha \xi_1) \tag{3.33}$$

then

$$e^{-x/\alpha} \Rightarrow e^{-\frac{1}{2}(\xi_0/\alpha + \xi_1)} \tag{3.34}$$

The point of the examples given above is to make clear a fundamental issue: the behavior in α is drastically affected by extension. The extension must be chosen in the manner most convenient for whatever problem is at hand.

C. Application of the Concept of Extension to Uniformly Valid Approximations

We consider now a class of mappings $\{f\}$ with domain D and range R. Thus,

$$f: \; D \longrightarrow R \tag{3.35}$$

is a function of physical interest, its domain and range are therefore defined by detailed consideration of the problem at hand. For example, if we are concerned with the Liouville equation, the class $\{f\}$ is the class $\{F^N\}$ of (1.3), the domain D is the N-body phase space and the range R is the set of positive real numbers.

We now apply the concept of extension to both D and R i.e., we identify δ of part A of this section, with D on the one hand and with R on the other (see table I).

We therefore consider one to one invertible mappings of both the domain and the range. Thus,

$$E_D : \quad D \rightarrow I_D \tag{3.36}$$

$$E_R : \quad R \rightarrow I_R \tag{3.37}$$

We further embed I_D and I_R in regions of sufficiently high dimension $\underline{D}$ and $\underline{R}$ which we call the extensions of the domain and of the range respectively:

$$I_D \subset \underline{D} \tag{3.38}$$

$$I_R \subset \underline{R} \tag{3.39}$$

We also consider a region, R^1, such that

$$R \subset R' \tag{3.40}$$

and the class of mappings $\mathcal{F}$ of R into R^1

$$\mathcal{F} : \quad R \rightarrow R' \tag{3.41}$$

We furthermore consider four "star" regions, D^*, R^*, R^{**}, R^{1*}, such that:

$$D \subset D^* \tag{3.42}$$

$$R \subset R^* \tag{3.43}$$

$$R \subset R^{**} \tag{3.44}$$

$$R' \subset R'^* \tag{3.45}$$

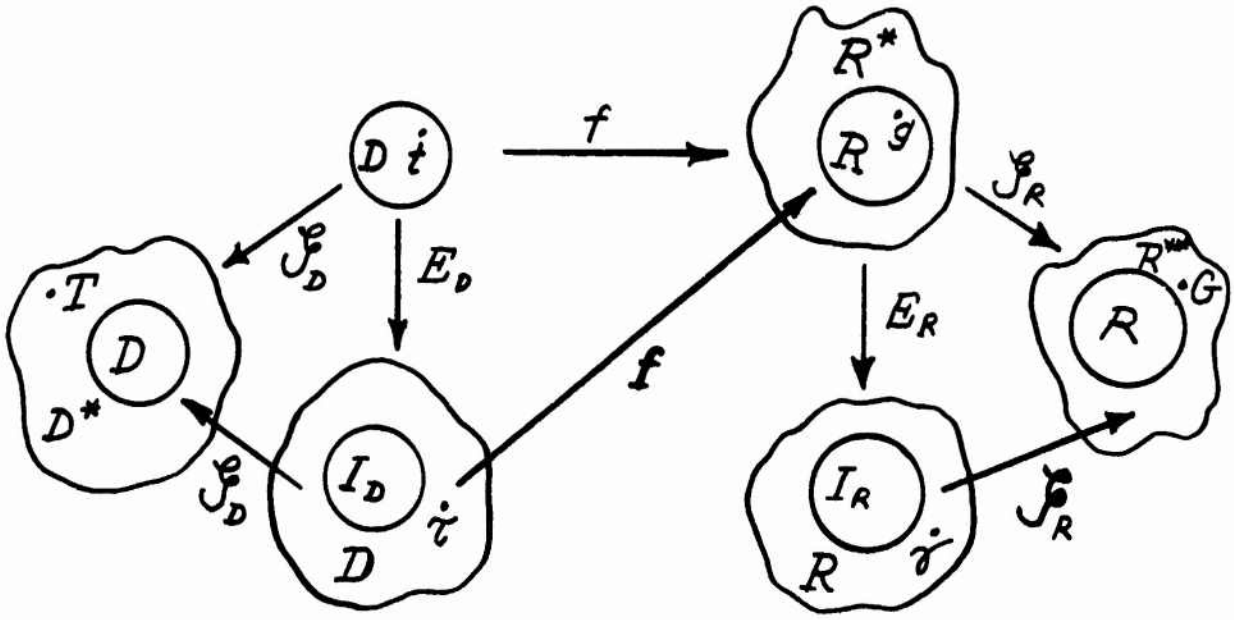

Fig. 5.

We use both the identity mappings for D and R:

$$\mathcal{J}_D : \quad D \longrightarrow D \tag{3.46}$$

$$\mathcal{J}_R : \quad R \longrightarrow R \tag{3.47}$$

We now consider the extensions of f and $\mathcal{J}$. That is, by (3.6), $\underline{f}$ extends f if and only if

$$\underline{f} : \quad \underline{D} \longrightarrow R^* \tag{3.48}$$

and

$$\underline{f} \circ E_D = f \tag{3.49}$$

Also, $\underline{\mathcal{F}}$ extends $\mathcal{F}$ if and only if

$$\underline{\mathcal{F}}: \underline{R} \longrightarrow R'^{*} \tag{3.50}$$

and

$$\mathcal{F} \circ E_R = \mathcal{F} \tag{3.51}$$

In particular, we extend the identity mappings. $\underline{\mathcal{J}_D}$ extends $\mathcal{J}_D$ if and only if

$$\underline{\mathcal{J}_D}: \underline{D} \longrightarrow D^{*} \tag{3.52}$$

and

$$\underline{\mathcal{J}_D} \circ E_D = \mathcal{J}_D \tag{3.53}$$

Also, $\underline{\mathcal{J}_R}$ extends $\mathcal{J}_R$ if and only if

$$\underline{\mathcal{J}_R}: \underline{R} \longrightarrow R^{**} \tag{3.54}$$

and

$$\underline{\mathcal{J}_R} \circ E_R = \mathcal{J}_R \tag{3.55}$$

$x \in \delta,\ X \in \delta^{*}$	$t \in D,\ T \in D^{*}$	$g \in R,\ G \in R^{**}$
$I\delta \subset \underline{\delta}$	$I_D \subset \underline{D}$	$I_R \subset \underline{R}$
$y \in \rho,\ Y \in \rho^{*}$	$g \in R$	$g' \in R'$
$\delta \xrightarrow{\phi} \rho$	$D \xrightarrow{f} R$	$R \xrightarrow{\mathcal{J}_R} R$
$\delta \xrightarrow{E_\delta} I_\delta \subset \underline{\delta}$	$D \xrightarrow{E_D} I_D \subset \underline{D}$	$R \xrightarrow{E_R} I_R \subset \underline{R}$
$\underline{\delta} \xrightarrow{\Phi} \rho^{*}$	$\underline{D} \xrightarrow{f} R^{*}$	$\underline{R} \xrightarrow{\mathcal{J}_R} R^{**}$

Table I

We clearly have the two induction theorems:

$$f \circ [\underline{\mathcal{I}}_D]_{I_D} = [\underline{f}]_{I_D} \tag{3.56}$$

and

$$\mathcal{F} \circ [\underline{\mathcal{I}}_R]_{I_R} = [\underline{\mathcal{F}}]_{I_R} \tag{3.57}$$

We furthermore introduce points by the following definitions

$$\{t\} = D, \quad \{\tau\} = \underline{D}, \quad \{T\} = D^* \tag{3.58}$$

$$\{g\} = R, \quad \{\gamma\} = \underline{R}, \quad \{G\} = R^{**}, \quad \{g'\} = R'^* \tag{3.59}$$

Local embedding equations in $\underline{D}$ and $\underline{R}$ are respectively

$$\text{at } \tau: \quad \tau_i = \tau_i(s) \quad \text{where } s = s(t) \tag{3.60}$$

$$\text{at } \gamma: \quad \gamma_i = \gamma_i(h) \quad \text{where } h = h(g) \tag{3.61}$$

In addition we must consider the extension of functionals. We have, as before,

$$f: \quad D \longrightarrow R \tag{3.35}$$

consider now a functional, such as a differential or integro-differential operator, Δ :

$$\Delta: \quad f \longrightarrow f^* \tag{3.62}$$

If f is extended by:

$$f \Longrightarrow \underline{f} \tag{3.63}$$

we shall say that $\underline{\Delta}$ is an extension of Δ if and only if, upon restriction:

$$\underline{\Delta}: \quad \underline{f} \longrightarrow f^* \tag{3.64}$$

Of course, a definite range is presupposed for $\underline{\Delta}$.

The broad aim of extension is to facilitate the understanding of the usually intricate behavior of a function of small parameter especially where a direct expansion fails. In J. R. Oppenheimer's words, extension is a method for "opening" a problem:

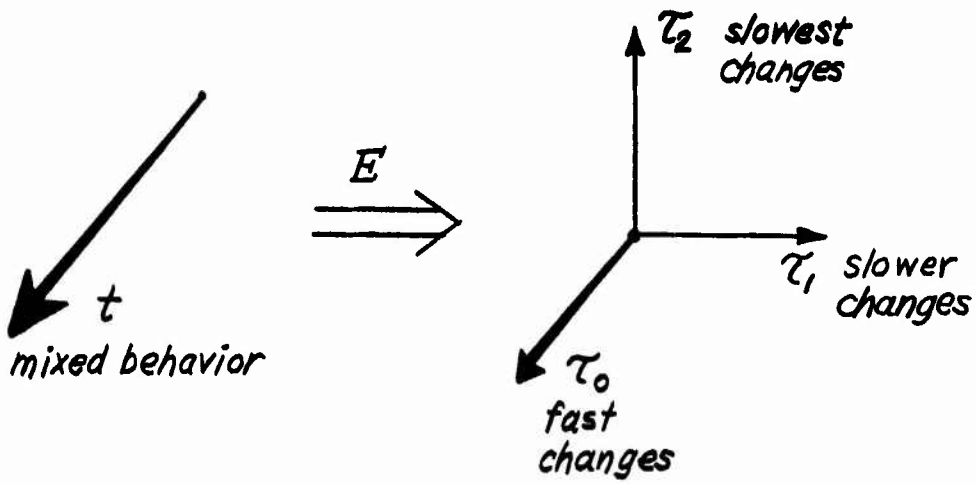

Fig. 6.

D. Examples of Extension

(1) Perturbation Theory. We begin with a familiar situation which is an extension of the range. We set, for a scalar valued

function g

$$g(t) = g_0(t) + \alpha g_1(t) + \alpha^2 g_2(t) + \dots \tag{3.65}$$

If we now choose as embedding coordinates

$$\gamma_0 = g, \ \gamma_1 = g/\alpha, \dots, \gamma_n = g/\alpha^n, \dots \tag{3.66}$$

then the image of R is very simply parametrized by g. We further set

$$g_i = a_i \gamma_i \tag{3.67}$$

with

$$\sum_{i=0}^{\infty} a_i = 1 \tag{3.68}$$

and

$$G(\gamma_0, \gamma_1, \dots, \gamma_n, \dots) = \sum_{i=0}^{\infty} a_i \alpha^i \gamma_i \tag{3.69}$$

Note that by convention superscripts are powers. We then have

$$[G]_{I_R} = \sum_{i=0}^{\infty} a_i \frac{g_i}{a_i} \alpha^i = g \tag{3.70}$$

as desired.

Even for very elementary perturbation expansions, the situation is slightly more complex. Thus, if

$$g(t) = e^{-\alpha t} \qquad 0 \le t < \infty, \ \alpha < 1 \tag{3.71}$$

Then $t = \frac{1}{\alpha} \log g$ and from the series:

$$g(t) = 1 - \alpha t + \frac{\alpha^2 t^2}{2} - \dots = g_0 + \alpha g_1 + \alpha^2 g_2 + \dots \qquad (3.72)$$

we find

$$g_0 = 1,\ g_1 = \frac{1}{\alpha} \log g, \dots, g_n = \frac{(-1)^n}{n!} \frac{1}{\alpha^n} \log^n g, \dots \qquad (3.73)$$

or, with (3.67) and (3.68)

$$\gamma_i = \frac{1}{a_i} \frac{(-1)^i}{i!} \frac{1}{\alpha^i} \log^i g \qquad (3.74)$$

Substituting into (3.69)

$$[G(\gamma_0, \gamma_1, \dots, \gamma_n, \dots)]_{I_R} = \sum_{i=0}^{\infty} \alpha^i \left\{ \frac{(-1)^i}{i! \alpha^i} \log^i g \right\} = e^{-\alpha t} \qquad (3.75)$$

A much more imaginative ansatz than (3.65) or (3.69) is used for nonlinear oscillations. Thus, let

$$g(t) = A(t) \sin B(t) \qquad (3.76)$$

with

$$B(t) = \omega t + b \qquad (3.77)$$

Power series for A and B (or b) are then used. The extension

of the range here is not linear. That is,

$$G(\gamma_0, \gamma_1, \ldots \gamma_n, \ldots) = \underline{A}(\gamma) \sin \underline{B}(\gamma) \tag{3.78}$$

with

$$[\underline{A}]_{I_R} = A, \quad [\underline{B}]_{I_R} = B \tag{3.79}$$

whence

$$[G]_{I_R} = g(t) \tag{3.80}$$

Nemytskii and Stepenov (4) use what amounts to a polynomial extension of the range to analyse solutions near singular points.

(ii) The Poincaré-Lighthill method. Here we choose a linear extension of the identity mapping on the domain whose values "restrict" to t. Thus, let

$$[T(\tau_0, \tau_1, \ldots, \tau_n, \ldots)]_{I_D} = t \tag{3.81}$$

that is we have a linear function $T(\tau_0, \tau_1, \ldots \tau_n \ldots)$ which, when

$$\tau_i = \tau_i(s) \tag{3.82}$$

$$s = s(t) \tag{3.83}$$

is equal to t. Thus,

$$T(\tau_0, \tau_1, \ldots \tau_n, \ldots) = \sum_i h_i(\alpha) \tau_i(s) \tag{3.84}$$

where $h_{i+1}/h_i \longrightarrow 0$ as $\alpha \rightarrow 0$. On I_D, (3.84) yields the famous expansion of the independent variable

$$t - \sum_i h_i(\alpha)\tau_i(s) \tag{3.85}$$

Since

$$\frac{df}{dt} = \frac{df}{ds}\frac{ds}{dt} \tag{3.86}$$

We obtain from (3.85)

$$\frac{df}{ds} = \frac{df}{dt}\sum_i h_i(\alpha)\frac{d\tau_i(s)}{ds} \tag{3.87}$$

This formula constitutes an expansion of the derivative which, as we shall see, is completely different from the Enskog-Bogolubov expansion.

A special case of the Poincare-Lighthill extension is obtained with a regular embedding sequence of the type (3.19). In fact, if (3.83) reads

$$s = t \tag{3.88}$$

and if we choose

$$\tau_i = a_i \frac{t}{h_i(\alpha)} \quad , \quad \sum_{i=0}^{\infty} a_i = 1 \tag{3.89}$$

then, from (3.84) we have, on I_D

$$[T]_{I_D} = \sum_{i=0}^{\infty} h_i(\alpha)\left\{a_i \frac{t}{h_i(\alpha)}\right\} = t \tag{3.90}$$

(iii) The Enskog-Bogolubov expansion of the time derivative. We consider the extension

$$f(t) \Longrightarrow \underline{f}(\tau_0, \tau_1, \ldots \tau_n, \ldots) \tag{3.91}$$

and a singular sequence of the type (3.17)

$$\tau_0 = s, \; \tau_1 = h_1(\alpha)s, \ldots \tau_n = h_n(\alpha)s, \ldots \tag{3.92}$$

We now take

$$\frac{\partial f}{\partial s} \Longrightarrow \sum_n h_n(\alpha) \frac{\partial \underline{f}}{\partial \tau_n} \tag{3.93}$$

The kinetic technique in effect averages successively over τ_0, $\tau_1, \ldots \tau_n \ldots$. This will be discussed in more detail later because it leads to an important equivalence theorem:

<u>The Synchronization Assumption of Bogolubov is Equivalent to the Time-Averaging of Kirkwood</u>. A special case of (3.93) has been independently considered by E. Frieman (<u>5</u>). Other related works are listed in (<u>5</u>).

SECTION 4

SUMMARY OF FORMULAE FOR EXTENSION

The Concept of Extension

Consider the mapping $\phi : \delta \rightarrow \rho$. Extend its domain $E_\delta : \delta \rightarrow I_\delta \subset \underline{\delta}$. Let local coordinates on $\underline{\delta}$ be given by $\xi_i = \xi_i(\lambda)$. Extend the mapping:

$$\phi \Rightarrow \underline{\phi} \;.\equiv.\; \underline{\phi} \circ E_\delta = \phi$$

Induction Theorem

Consider the identity mapping $\mathcal{I}_\delta : \delta \rightarrow \delta$. Then $\mathcal{I}_\delta \Rightarrow \underline{\mathcal{I}_\delta}$ gives $\phi = \phi \circ \underline{\mathcal{I}_\delta}$ on I_δ. Note $[\mathcal{I}_\delta]_{I_\delta} = E_\delta^{-1}$

Application to Uniform Expansion in a Small Parameter

Consider $f: D \rightarrow R$, $\mathcal{F} : R \rightarrow R'$. Extend D and R: $E_D: D \rightarrow I_D$, $E_R: R \rightarrow I_R$. Extend the functions: $f \Rightarrow \underline{f} \;.\equiv.\; \underline{f} \circ E_D = f$. Induction formulae:

$$\mathcal{F} \Rightarrow \underline{\mathcal{F}} \;.\equiv\; \underline{\mathcal{F}} \circ E_R = \mathcal{F}$$

$$\mathcal{I}_D \Rightarrow \underline{\mathcal{I}_D} \quad \text{then} \quad \underline{f} = f \circ \underline{\mathcal{I}_D} \text{ on } I_D$$

$$\mathcal{I}_R \Rightarrow \underline{\mathcal{I}_R} \quad \text{then} \quad \underline{\mathcal{F}} = \mathcal{F} \circ \underline{\mathcal{I}_R} \text{ on } I_R$$

We have $\left[\underline{\mathcal{I}_D}\right]_{I_D} = E_D^{-1}$, $\left[\underline{\mathcal{I}_R}\right]_{I_R} = E_R^{-1}$

The interrelation of the mappings and points is as follows:

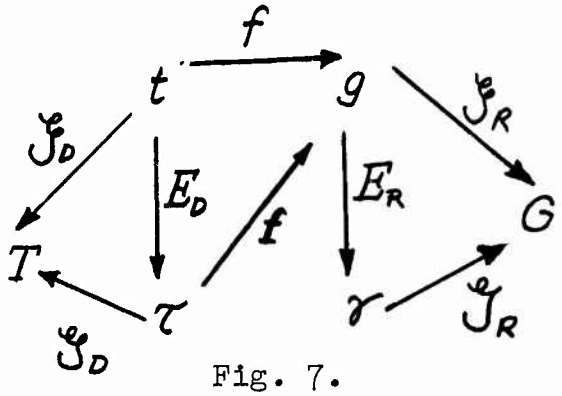

Fig. 7.

SECTION 5

THE MODIFIED UHLENBECK DIAGRAM

Before entering the details of the theory we summarize our views by means of the following modification of the Uhlenbeck diagram

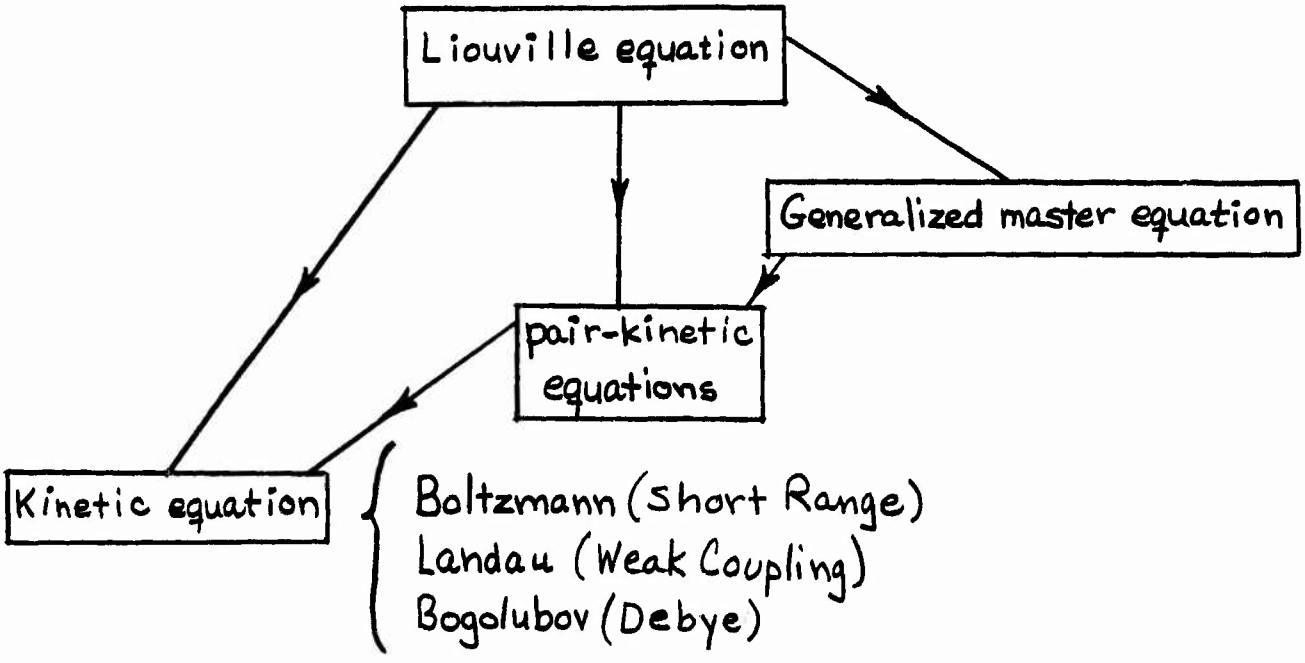

Fig. 8.

The detailed meaning of the terms will become clear as we proceed. We will prove that there are gas dynamic regimes for which a kinetic description is impossible. By means of appropriate generalizations of the Enskog-Bogolubov expansion technique we shall obtain equa-

tions valid for such regimes. An analogous situation occurs when the transition into the regime of fluid dynamics is considered. This theory is however considerably less developed (cf. Section 15). The quantum mechanical theory also awaits development. From the discussion of Section 3, it is clear that extension is a very broad framework with which a great many problems can be approached.

An example of an expansion which is "intermediate" between the Poincare-Lighthill and the Enskog-Bogolubov techniques is discussed by means of a single model for the Navier-Stokes equations (6).

SECTION 6

LIOUVILLE EQUATION - KINETIC PARAMETERS

The purpose of this section is to set up the appropriate expansion scheme that allows for the discussion of the kinetic equations (defined precisely at the end of this section) starting from the first principles represented by the Liouville equation.

Kirkwood has in part achieved this scope by his time averaging method (7), but his theory failed completely in giving the errors committed (e.g. it was impossible with Kirkwood's theory to obtain the corrections to Boltzmann's equation).

Bogolubov made considerable progress in giving a scheme that distinguishes gases with neutral elementary constituents from ionized gases from the start. His theory was thought also capable of giving the corrections to the Boltzmann equation (3) (3-body effects) as well as to the newly obtained kinetic equation for plasmas (8) (Bogolubov-Balescu-Lenard equation). The hope of obtaining correctly the three-body effects with Bogolubov's expansion is in fact unfounded (see Section 11).

Bogolubov's theory fails in two major respects: a. It fails to give finite (i.e. represented by convergent integrals) corrections to the kinetic equation. b. There is no possiblity to determine the conditions of validity of the lowest approximation. Both problems will be discussed and solved in the sequel. First of all, we have solved the problem of determining the conditions under which kinetic equations hold. The requirement that a gas should approach thermodynamical equilibrium via a kinetic regime defines sharply the class of correlation functions whose presence at $t = 0$ will be "forgotten" so as to permit the contraction:

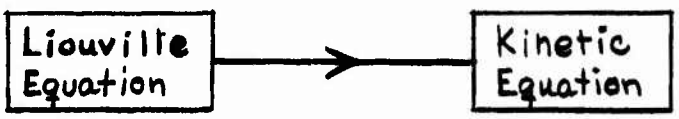

Fig. 9.

The class of these correlations will be called kinetic. The separation of these two classes leads to an important new principle in kinetic theory--the principle of absence of parallel motions (Section 8).

Secondly, we shall prove a very general phenomenon: to a sufficiently high order in the (Bogolubov) expansion that leads to the kinetic equations there is a divergence due to the extreme persistence of correlations (with a correlation length comparable to a mean free path) which arises because of successive two-body collisions. The breakdown occurs in second order for the short range expansion and for the Debye expansion, in fourth order in a weakly-coupled gas and in fifth order for a dilute weakly coupled gas. The breakdown however is not alarming because the expansions carried out are asymptotic.

If we want to go beyond the kinetic theory therefore we must devise a "weaker" limiting procedure. The key concept in this theory is the concept of closure (Section 9).

This concept has been made possible only by a full exploitation of the method of extension. With this method we can in fact select solutions of the Liouville equation with a desired degree of "simplicity" of approach to equilibrium. The simplest choice is the kinetic. It will be clear that there is a complete hierarchy of increasingly more informative solutions of the Liouville equation. It will also be clear that the amount of information obtained at a particular level of "complexity", if the approximation is carried out to the point of closure, is statistically complete. Thus, in our view, kinetic equations (which do satisfy closure) are exact.

At the highest level of complexity, one finds the generalized master equations (Section 13) which determine the probability distribution of the momenta of all the particles in the system.

The starting point of the theory is the Liouville equation (9)

for the N-body distribution function F^N:

$$\frac{dF^N}{dt} = 0 \tag{6.1}$$

F^N depends on time and on the 2N vectors $\underline{x}_i$ and $\underline{p}_i = M\underline{v}_i$ which define the position and momenta of the particles (of mass $m=1$) in phase space. Since the N masses which interact through the two-body potential, U_{ij}, satisfy Newton's laws of motion:

$$\dot{\underline{x}}_i = \underline{v}_i \,, \qquad \dot{\underline{p}}_i = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\partial U_{ij}}{\partial \underline{x}_i} \tag{6.2}$$

we can rewrite (6.1) as:

$$\frac{\partial F^N}{\partial t} + \sum_{i=1}^{N} \frac{\partial F^N}{\partial \underline{x}_i} \cdot \underline{v}_i - \sum_{1\le i<j\le N} I_{ij} \cdot F^N = 0 \tag{6.3}$$

where I_{ij} is the interaction operator:

$$I_{ij} = \frac{\partial U_{ij}}{\partial \underline{x}_i} \cdot \frac{\partial}{\partial \underline{v}_i} + \frac{\partial U_{ij}}{\partial \underline{x}_j} \cdot \frac{\partial}{\partial \underline{v}_j} \tag{6.4}$$

It is convenient to introduce also the kinetic energy operators:

$$K_i \equiv \underline{v}_i \cdot \frac{\partial}{\partial \underline{x}_i} \qquad i = 1, 2, \ldots, N \tag{6.5}$$

and the total energy operators:

$$\mathcal{H}^S = K^S - I^S = \sum_{i=1}^{S} K_i - \sum_{1 \le i < j \le S} I_{ij} \tag{6.6}$$

where S has the range

$$1 \le S \le N \tag{6.7}$$

With this notation (6.3) acquires the convenient form

(1.3)
$$\frac{\partial F^N}{\partial t} + \mathcal{H}^N F^N = 0 \tag{6.8}$$

We take F^N to be normalized to one by

$$\int F^N \frac{d\underline{x}_1}{V} d\underline{V}_1 \frac{d\underline{x}_2}{V} d\underline{V}_2 \cdots \frac{d\underline{x}_N}{V} d\underline{V}_N = 1 \tag{6.9}$$

where V is the volume of the box enclosing the gas.

If we now introduce the distribution functions

$$F^S = \int F^N \frac{d\underline{x}_{S+1}}{V} d\underline{V}_{S+1} \cdots \frac{d\underline{x}_N}{V} d\underline{V}_N \tag{6.10}$$

we find that they satisfy

$$\frac{\partial F^S}{\partial t} + \mathcal{H}^S F^S = \frac{N-S}{V} L_S F^{S+1} \tag{6.11}$$

where

$$L_s = \sum_{i=2}^{s} L_{1i}\,,\quad L_{1i} = \int d\underline{x}_i\, d\underline{v}_i\, \underline{\nabla}_1 U_{1i} \cdot \underline{\nabla}_{v_1} \tag{6.12}$$

Since we are particularly interested in the properties of a gas where both N and V are large but for which the mean density is a given number:

$$n = \frac{N}{V} \qquad \text{given} \tag{6.13}$$

we consider only the limit of (6.11) as $N \to \infty$ and $V \to \infty$ subject to (6.13).

The resulting hierarchy of equations (Bogolubov 1946, Born and Green, Kirkwood, Yvon (10)) is:

$$\frac{\partial F^s}{\partial t} + \mathcal{H}^s F^s = n L_s F^{s+1} \tag{6.14}$$

which, in contrast to the Liouville equation (6.8) make explicit the role of the mean density. We want to emphasize, however, also the role of the strength ϕ_0 of the two-body interaction. We make, accordingly, the hierarchy (6.14) dimensionless with respect to the kinetic temperature $m v_{th}^2$, the range r_0 and the strength ϕ_0 of the two-body interaction. We thus obtain remembering (6.6) and also (6.9) and (6.10):

$$\frac{\partial F^s}{\partial t} + \mathcal{K}^s F^s - \left(\frac{\phi_0}{m v_{th}^2}\right) I^s F^s = \left(n r_0^3\right)\left(\frac{\phi_0}{m v_{th}^2}\right) L_s F^{s+1} \tag{6.15}$$

where all the quantities are now dimensionless (we do not employ new symbols because no confusion can arise in the sequel). The equation (6.15) has the advantage over (6.14) or (6.8) of making

explicit two very important parameters of the theory of gases nr_0^3 and ϕ_0/mv_{th}^2 . In terms of these parameters, the basic regimes of gas theory are defined as follows:

(i) Weakly-coupled gas

$$nr_0^3 \simeq 1 \,, \quad \frac{\phi_0}{mv_{th}^2} << 1 \tag{6.16}$$

(ii) Dilute gas with strong short range forces (short-range regime)

$$nr_0^3 << 1 \,, \quad \frac{\phi_0}{mv_{th}^2} \simeq 1 \tag{6.17}$$

(iii) Gas with Coulomb forces in the "Debye regime"

$$nr_0^3 = \frac{1}{\epsilon} \,, \quad \frac{\phi_0}{mv_{th}^2} = \epsilon << 1 \tag{6.18}$$

(iv) Dilute weakly-coupled gas (DWC)

$$nr_0^3 = \frac{\phi_0}{mv_{th}^2} = \epsilon << 1 \tag{6.19}$$

The various regimes are best visualized by means of a diagram:

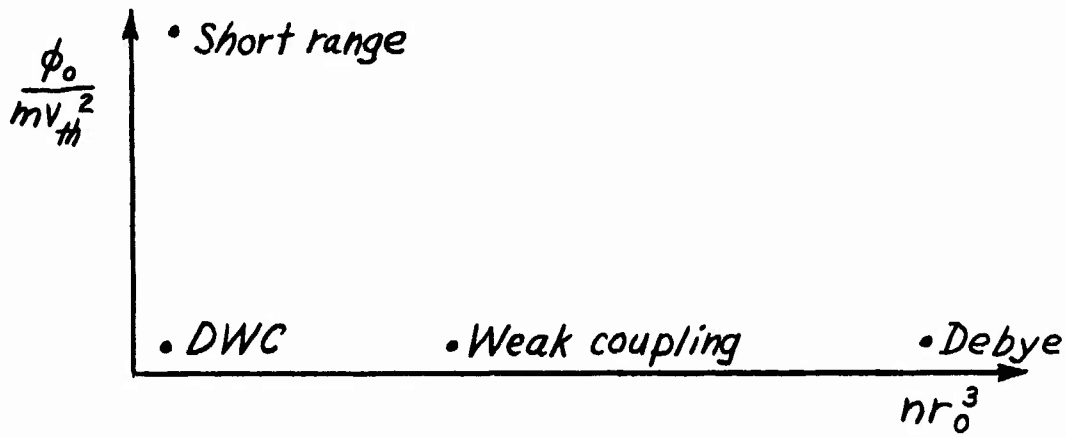

Fig. 10.

All the points of interest lie close to one of the axes. If one wants to expand, something has to be small! If one eliminates ϵ between the two relations in (iii) for the Debye regime one finds

$$r_0 = \sqrt{\frac{e^2 n}{m k v_{th}^2}} = \lambda_{Debye} \tag{6.20}$$

We now define precisely the concept of kinetic equation. From (6.15) we have, for the one particle distribution function; putting $s = 1$

$$\frac{\partial F^1}{\partial t} + K_1 F^1 = (n r_0^3)\left(\frac{\phi_0}{m v_{th}^2}\right) L F^2 \tag{6.21}$$

where, from (6.10)

$$F^1 = \int F^N d\Gamma_{-1} \tag{6.22}$$

where $d\Gamma_{-1}$ denotes integration over all of the phase space coordinates with the exception of $\underline{X}_1$ and $\underline{V}_1$.

We say that the gas is in a kinetic regime if the one-body distribution function satisfies an equation of the form

$$\frac{\partial F^1}{\partial t} = A\,[F^1] \tag{6.23}$$

where $A\,[F^1]$ is any functional of F^1 only, that is to say if

$$(n r_0^3)\left(\frac{\phi_0}{m v_{th}^2}\right) L F^2 = A[F^1] + K_1 F^1 \tag{6.24}$$

The outstanding examples of kinetic equations are the Boltzmann equation and the Fokker-Planck equation. The Boltzmann equation is

$$\frac{\partial F^1}{\partial t} + \mathcal{K}_1 F^1 = \int d\underline{V}_2 \, \sigma g \, d\Omega \left[F_{1*}^1 F_{2*}^1 - F_1^1 F_2^1 \right] \qquad (6.25)$$

where $g = |\underline{V}_1 - \underline{V}_2|$, σ is the two-body scattering cross-section, $d\Omega$ the solid angle of scattering and the subscripts on the F^1, function denote the velocities in a two-body encounter:

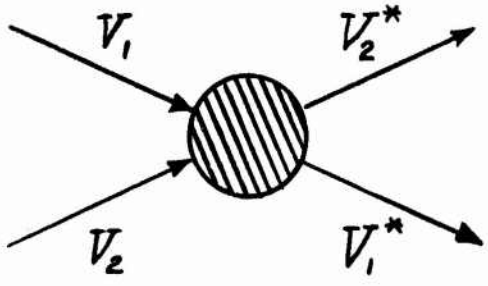

Fig. 11.

The positions on the right hand side of (6.25) are taken to coincide with that of body 1:

$$\underline{X}_i \approx \underline{X}_1 \qquad (6.26)$$

The Fokker-Planck equation reads:

$$\frac{\partial F^1}{\partial t} + \mathcal{K}_1 F^1 = \frac{\partial}{\partial \underline{V}_1} \cdot \left(\underline{A} F^1 \right) + \frac{\partial^2}{\partial \underline{V}_1 \partial \underline{V}_1} : \left(\underline{\underline{B}} F^1 \right) \qquad (6.27)$$

where $\underline{A}$ is the coefficient of dynamical friction and $\underline{\underline{B}}$ the coefficient for diffusion in velocity space.

SECTION 7

WEAK-COUPLING EXPANSION

In view of (6.16) we can write the hierarchy (6.15) as

$$\frac{\partial F^s}{\partial t} + K^s F^s = \varepsilon I^s F^s + \varepsilon L_s F^{s+1} \tag{7.1}$$

We shall consider a spatially homogeneous gas excepting Section 15.

We now introduce the expansions

$$F^1 = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \varepsilon^3 f^3 + \varepsilon^4 f^4 + \cdots \tag{7.2}$$

$$F^2 = F^{20} + \varepsilon F^{21} + \varepsilon^2 F^{22} + \varepsilon^3 F^{23} + \cdots \tag{7.3}$$

$$F^3 = F^{30} + \varepsilon F^{31} + \varepsilon^2 F^{32} + \cdots \tag{7.4}$$

$$F^4 = F^{40} + \varepsilon F^{41} + \cdots \tag{7.5}$$

$$F^5 = F^{50} + \cdots \tag{7.6}$$

It will be seen that the reason for stopping the expansion at the places indicated is the breakdown of the kinetic behavior.

Convention for the Notation. Superscripts represent the number of particles in the cluster considered, subscripts denote the phase points at which a function is taken. There is never need for both since, for example:

$$F_{134} \equiv F^3(\underline{X}_1, \underline{V}_1, \underline{X}_2, \underline{V}_2, \underline{X}_3, \underline{V}_3) \tag{7.7}$$

unambiguously. When no confusion arises, subscripts are omitted. Thus,

$$f^\circ f^\circ \equiv f_1^\circ f_2^\circ \tag{7.8}$$

A. Perturbation Theory - Choice of Embedding Coordinates

If we perform the direct perturbation expansion, we have in the limit $\epsilon \to 0$

$$\frac{\partial F^{s0}}{\partial t} + \mathcal{K}^s F^{s0} = 0 \tag{7.9}$$

for $s > 0$ and

$$\frac{\partial f^\circ}{\partial t} = 0 \tag{7.10}$$

The general solution of (7.9) is

$$F^{s0}(t) = e^{-\mathcal{K}^s t} F^{s0}(0) \tag{7.11}$$

We now introduce the correlation functions C^s by the definition:

$$F^s = \frac{s}{\pi} F^1 + C^s \tag{7.12}$$

We introduce the notion of "simple initial value problem" by requiring

$$C^s(t=0) = 0 \tag{7.13}$$

$$\left[\frac{\partial F^1}{\partial \varepsilon}\right]_{t=0} = 0 \tag{7.14}$$

In contrast, for the "complete initial value" problem there are no restrictions on the initial values of the correlations or of the one-body distribution.

Let us confine ourselves to the simple initial value problem. From (7.11):

$$F^{s0}(t) = e^{-K^s t} \overset{s}{\Pi} f^0 \tag{7.15}$$

In particular for s = 2

$$F^{20}(t) = e^{-K^s t} f^0 f^0 \tag{7.16}$$

Since the gas is spatially homogeneous:

$$K_1 f_1^0 = 0 \,, \quad K_2 f_2^0 = 0 \tag{7.17}$$

and therefore

$$F^{20}(t) = F^{20}(0) = f^0 f^0 \tag{7.18}$$

From (7.1), in first order

$$\frac{\partial f^1}{\partial t} = L F^{20} = L f^0 f^0 \tag{7.19}$$

But we have

$$\begin{aligned} \mathcal{L} f^0 f^0 &= \int d\underline{x}_2\, d\underline{v}_2\, \underline{\nabla}_1 U_{12} \cdot \underline{\nabla}_{v_1} f_1^0 f_2^0 \\ &= \int d\underline{x}_2\, d\underline{v}_2\, U_{12} \underline{\nabla}_{v_1} \cdot \underline{\nabla}_2 f_1^0 f_2^0 \\ &= 0 \end{aligned} \tag{7.20}$$

where we have exploited the translation invariance of U_{12} and then our ability to perform integration by parts neglecting the surface terms at infinity (wall potential). Again, from (7.1), in first order for the two-body function:

$$\frac{\partial F^{21}}{\partial t} + K^2 F^{21} = I_{12} F^{20} + \mathcal{L}_2 F^{30} \tag{7.21}$$

but again

$$\mathcal{L}_2 f^0 f^0 f^0 = 0$$

therefore

$$\begin{aligned} F^{21}(t) &= \frac{1 - e^{-K^2 t}}{K^2} I^2 f^0 f^0 \\ &= \int_0^t e^{-K^2 \lambda}\, d\lambda\, I^2 f^0 f^0 \end{aligned} \tag{7.22}$$

We then find, from (7.1), for the second order one-body function:

$$\frac{\partial f^2}{\partial t} = \mathcal{L} F^{21} = \mathcal{L} \int_0^t e^{-K^2 \lambda}\, d\lambda\, I^2 f^0 f^0 \tag{7.23}$$

We note now that for large t, (7.22) gives

$$F^{21}(t) \underset{t}{\sim} \int_0^\infty e^{-k^2\lambda} d\lambda I^2 f^0 f^0 = \zeta^*(ik^2) I^2 f^0 f^0 \tag{7.24}$$

where we have used the notation:

$$\zeta^*(ik^2) = \int_0^\infty e^{-k^2\lambda} d\lambda \equiv \zeta^*_{12} \tag{7.25}$$

We shall use also

$$\delta(ik^2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-k} \tag{7.26}$$

Using (7.24) in (7.23):

$$f^2(t) = f^2(0) + \int_0^t dt' L \int_0^{t'} e^{-k^2\lambda} d\lambda I^2 f^0 f^0$$
$$\underset{t}{\sim} t L(\zeta^* I)_{12} f^0 f^0 \tag{7.27}$$

where we have used the fact that for the simple initial value problem

$$f^2(0) = 0$$

We have therefore

$$\frac{\epsilon^2 f^2}{f^0} = (\epsilon^2 t) \frac{L \zeta^* I f^0 f^0}{f^0} \tag{7.28}$$

Since the quantity in square brackets is time independent, the

approximation breaks down when

$$t \gtrsim \frac{1}{\epsilon^2}$$

In particular the normalization of F^1 will be ruined when this condition is fulfilled. We will therefore perform an extension of the functions F^s in order to remove this defect of the direct expansion. We need a means of choosing the embedding coordinates. The perturbation expansion breaks down with a simple linearity in (7.28). We choose therefore our new coordinates in such a way as to be able to follow this behavior (11). Thus,

$$\tau_0 = t, \ \tau_1 = \epsilon t, \ldots \tau_n = \epsilon^n t, \ldots \qquad (7.29)$$

We shall use boldface to indicate the extended functions. For the extension of the range we use simply perturbation theory. We have accordingly

$$\frac{\partial}{\partial t} \Rightarrow \sum \epsilon^k \frac{\partial}{\partial \tau_k} \qquad (7.30)$$

$$f^k \Longrightarrow \underline{f}^k, \quad F^{sk} \Longrightarrow \underline{F}^{sk} \qquad (7.31)$$

B. Uniform Expansion for the Simple Initial Value Problem

(i) Extended perturbation equations. Using the expansion (7.2) to (7.6), and the extensions (7.30) and (7.31) we obtain a sequence of "extended" perturbation equations. For the one-body function we have

$$\frac{\partial \underline{f}^0}{\partial \tau_0} = 0 \qquad (7.32)$$

$$\frac{\partial \underline{f}^{1}}{\partial \tau_0} + \frac{\partial \underline{f}^{0}}{\partial \tau_1} = L\underline{F}^{20} \tag{7.33}$$

$$\frac{\partial \underline{f}^{2}}{\partial \tau_0} + \frac{\partial \underline{f}^{1}}{\partial \tau_1} + \frac{\partial \underline{f}^{0}}{\partial \tau_2} = L\underline{F}^{21} \tag{7.34}$$

$$\frac{\partial \underline{f}^{3}}{\partial \tau_0} + \frac{\partial \underline{f}^{2}}{\partial \tau_1} + \frac{\partial \underline{f}^{1}}{\partial \tau_2} + \frac{\partial \underline{f}^{0}}{\partial \tau_3} = L\underline{F}^{22} \tag{7.35}$$

$$\frac{\partial \underline{f}^{4}}{\partial \tau_0} + \frac{\partial \underline{f}^{3}}{\partial \tau_1} + \frac{\partial \underline{f}^{2}}{\partial \tau_2} + \frac{\partial \underline{f}^{1}}{\partial \tau_3} + \frac{\partial \underline{f}^{0}}{\partial \tau_4} = L\underline{F}^{23} \tag{7.36}$$

For the two-body function:

$$\frac{\partial \underline{F}^{20}}{\partial \tau_0} + K^2 \underline{F}^{20} = 0 \tag{7.37}$$

$$\frac{\partial \underline{F}^{21}}{\partial \tau_0} + K^2 F^{21} = I^2 \underline{F}^{20} + L_2 \underline{F}^{30} - \frac{\partial \underline{F}^{20}}{\partial \tau_1} \tag{7.38}$$

$$\frac{\partial \underline{F}^{22}}{\partial \tau_0} + K^2 \underline{F}^{22} = I^2 \underline{F}^{21} + L_2 \underline{F}^{31} - \frac{\partial \underline{F}^{21}}{\partial \tau_1} - \frac{\partial \underline{F}^{20}}{\partial \tau_2} \tag{7.39}$$

$$\frac{\partial \underline{F}^{23}}{\partial \tau_0} + K^2 \underline{F}^{23} = I^2 \underline{F}^{22} + L_2 \underline{F}^{32} - \frac{\partial \underline{F}^{22}}{\partial \tau_1} - \frac{\partial \underline{F}^{21}}{\partial \tau_2} - \frac{\partial \underline{F}^{20}}{\partial \tau_3} \tag{7.40}$$

For the three-body distribution

$$\frac{\partial \underline{F}^{30}}{\partial \tau_0} + K^3 \underline{F}^{30} = 0 \tag{7.41}$$

$$\frac{\partial \underline{F}^{31}}{\partial \tau_0} + K^3 \underline{F}^{31} = I^3 \underline{F}^{30} - \frac{\partial \underline{F}^{30}}{\partial \tau_1} + L_3 \underline{F}^{40} \tag{7.42}$$

$$\frac{\partial \underline{F}^{32}}{\partial \tau_0} + K^3 \underline{F}^{32} = I^3 \underline{F}^{31} + L_3 \underline{F}^{41} - \frac{\partial \underline{F}^{31}}{\partial \tau_1} - \frac{\partial \underline{F}^{30}}{\partial \tau_2} \tag{7.43}$$

We also need, for the four- and five-body distributions:

$$\frac{\partial \underline{F}^{40}}{\partial \tau_0} + K^4 \underline{F}^{40} = 0 \tag{7.44}$$

$$\frac{\partial \underline{F}^{41}}{\partial \tau_0} + K^4 \underline{F}^{41} = I^4 \underline{F}^{40} + L_4 \underline{F}^{50} - \frac{\partial \underline{F}^{40}}{\partial \tau_1} \tag{7.45}$$

and

$$\frac{\partial \underline{F}^{50}}{\partial \tau_0} + K^5 F^{50} = 0 \tag{7.46}$$

we are solving the simple initial value problem. Therefore at $\tau_0 = 0$

$$\underline{f}^0(0) \text{ given} \tag{7.47}$$

$$\underline{f}^1(0) = \underline{f}^2(0) = \underline{f}^3(0) = \underline{f}^4(0) = 0 \tag{7.48}$$

for the one-body distribution. For the two-body distribution we have

$$\underline{F}^{20}(0) = \underline{f}^0(0)\,\underline{f}^0(0) \tag{7.49}$$

$$\underline{F}^{21}(0) = \underline{F}^{22}(0) = \underline{F}^{23}(0) = 0 \tag{7.50}$$

The three-body distribution satisfies:

$$\underline{F}^{30}(0) = \overset{3}{\Pi}\,\underline{f}^0(0) \tag{7.51}$$

$$\underline{F}^{31}(0) = F^{32}(0) = 0 \tag{7.52}$$

The four-body distribution satisfies

$$\underline{F}^{40}(0) = \overset{4}{\Pi}\,\underline{f}^0(0) \tag{7.53}$$

$$\underline{F}^{41}(0) = 0 \tag{7.54}$$

And finally

$$\underline{F}^{s0}(0) = \overset{5}{\pi}\, \underline{f}^{0}(0) \tag{7.55}$$

We remark that we do not give the phase space dependence of $\underline{F}^{s}$ but only the functional dependence of $\underline{F}^{s}(t_0 = 0)$ on $\underline{f}^{o}$. In the transition to hydrodynamics one will give analogously the functional dependence of $\underline{f}^{0}$ on the density, temperature, and flow velocity.

(ii) The Zeroth Order Theory

(7.32) insures that $\underline{f}^{0}$ is τ_0 independent i.e.:

$$\underline{f}^{0}(0) = \underline{f}^{0} \qquad \text{given} \tag{7.56}$$

(7.41), (7.44), and (7.46) can be summarized as

$$\frac{\partial F^{s0}}{\partial \tau_0} + K^{s}\underline{F}^{s0} = 0 \tag{7.57}$$

which can be solved immediately as

$$\underline{F}^{s0}(\tau_0) = e^{-K^{s}\tau_0}\underline{F}^{s0}(0) = e^{-K^{s}\tau_0}\overset{5}{\pi}\underline{f}^{0} = \overset{5}{\pi}\underline{f}^{0} \tag{7.58}$$

(iii) The First Order Theory

By the lemma (7.20)

$$L\,\underline{f}^{0}\underline{f}^{0} = 0 \tag{7.59}$$

Therefore (7.33) gives

$$\frac{\partial \underline{f}^{1}}{\partial \tau_0} + \frac{\partial \underline{f}^{0}}{\partial \tau_1} = 0 \tag{7.60}$$

or, upon integration

$$\underline{f}'(\tau_0) = \tau_0 \frac{\partial \underline{f}^0}{\partial \tau_1} \tag{7.61}$$

Therefore, if we require, for our asymptotic series

$$\frac{\epsilon \underline{f}'(\tau_0)}{\underline{f}^0} = (\epsilon \tau_0) \frac{\partial \underline{f}^0}{\partial \tau_1} \frac{1}{\underline{f}^0} = O(\epsilon) \quad \text{uniformly} \tag{7.62}$$

we must set,

$$\frac{\partial \underline{f}^0}{\partial \tau_1} = 0 \tag{7.63}$$

hence by (7.61)

$$\underline{f}' = 0 \tag{7.64}$$

For the two-body function we have, from (7.38):

$$\frac{\partial \underline{F}^{21}}{\partial \tau_0} + K^2 \underline{F}^{21} = I^2 \underline{f}^0 \underline{f}^0 \tag{7.65}$$

whence

$$\underline{F}^{21}(\tau_0) = \int_0^{\tau_0} e^{-K^2 \lambda} d\lambda \, I^2 \underline{f}^0 \underline{f}^0 \tag{7.66}$$

From (7.42) and (7.45) we also find similar formulae for $\underline{F}^{31}$, $\underline{F}^{41}$. In fact

$$\underline{F}^{51}(\tau_0) = \int_0^{\tau_0} e^{-K^5 \lambda} d\lambda \, I^5 \overset{5}{\Pi} \underline{f}^0 \tag{7.67}$$

The transient behavior of $\underline{F}^{21}$ on the τ_0 scale is given

by

$$\underline{F}^{21}(\tau_0) - \underline{F}^{21}_{asympt} = -\int_{\tau_1}^{\infty} e^{-\kappa^2\lambda}\, d\lambda\, I^2 \underline{f}\underline{f}^{\circ}$$

$$= -2\int_{\tau_0}^{\infty} \underline{\nabla}_{12}\, U(\underline{x}_{12} - \underline{v}_{12}\lambda)\, d\lambda \cdot \underline{\nabla}_{v_{12}} \underline{f}^{\circ}\underline{f}^{\circ}$$

The integral

$$\int_{\tau_0}^{\infty} \underline{\nabla}_{12}\, U(\underline{x}_{12} - \underline{v}_{12}\lambda)\, d\lambda$$

goes to zero sharply as τ_0 increases for fixed finite $\underline{v}_{12}$. It will be seen that the transients in F^1 behave very differently from those in F^2.

(iv) Second Order Theory (Landau Equation)

(7.34) gives with (7.66)

$$\frac{\partial \underline{f}^2}{\partial \tau_0} + \frac{\partial \underline{f}^{\circ}}{\partial \tau_2} = L\int_0^{\tau_0} e^{-\kappa^2\lambda}\, d\lambda\, I^2 \underline{f}^{\circ}\underline{f}^{\circ} \qquad (7.68)$$

Integration with respect to τ_0 yields:

$$\underline{f}^2(\tau_0) = L\int_0^{\tau_0} \frac{1 - e^{-\kappa^2\lambda}}{\kappa^2}\, d\lambda\, I^2 \underline{f}^{\circ}\underline{f}^{\circ} - \tau_0 \frac{\partial \underline{f}^{\circ}}{\partial \tau_2}$$

$$= -L\int_{\tau_0}^{\infty} \frac{1 - e^{-\kappa^2\lambda}}{\kappa^2}\, d\lambda\, I^2 \underline{f}^{\circ}\underline{f}^{\circ} - \qquad (7.69)$$

$$-\tau_0\left[\frac{\partial \underline{f}^{\circ}}{\partial \tau_2} - L\,\zeta^*(i\kappa^2)\, I^2 \underline{f}^{\circ}\underline{f}^{\circ}\right]$$

where the operator $\mathcal{S}^*(i\mathcal{K}^2)$ is given by (7.25). The requirement that $\epsilon^2 \underline{f}^2$ remains a small correction uniformly yields

$$\frac{\partial \underline{f}^\circ}{\partial \tau_2} = L\mathcal{S}^*(i\mathcal{K}^2) I^2 \underline{f}^\circ \underline{f}^\circ$$
$$= 2\pi i\, L\delta(i\mathcal{K}^2) I^2 \underline{f}^\circ \underline{f}^\circ \tag{7.70}$$

This is the kinetic equation of the Fokker-Planck type first obtained by Landau. The H-theorem is readily proved on the τ_2 time scale by the conventional symmetrization in the variables 1 and 2:

$$\frac{dH}{d\tau_2} = 2\iint d\underline{v}_1\, d\underline{v}_2 \frac{1}{f_1^\circ} \nabla_{v_1} \underline{f}_1^\circ \cdot \int \nabla_{12} U_{12}\, \delta(i\mathcal{K}_{12})\, \nabla_{12} U_{12}\, d\underline{x}_{12} \cdot$$
$$\cdot \left(\underline{f}_1^\circ \nabla_{v_2} \underline{f}_2^\circ - \underline{f}_2^\circ \nabla_{v_1} f_1^\circ\right) \tag{7.71}$$

$$= -\iint \frac{d\underline{v}_1 d\underline{v}_2}{\underline{f}_1^\circ f_2^\circ} \left(\underline{f}_1^\circ \nabla_{v_2} \underline{f}_2^\circ - \underline{f}_2^\circ \nabla_{v_1} \underline{f}_1^\circ\right) \cdot \int \nabla_{12} U_{12}\, \delta(i\mathcal{K}_{12}) \nabla_{12} U_{12}\, d\underline{x}_{12} \cdot$$
$$\cdot \left(\underline{f}_1^\circ \nabla_{v_2} \underline{f}_2^\circ - \underline{f}_2^\circ \nabla_{v_1} \underline{f}_1^\circ\right) \tag{7.72}$$

Substituting back (7.70) into (7.68) we see that $\underline{f}^2$ satisfies

$$\frac{\partial \underline{f}^2}{\partial \tau_0} = -L \int_{\tau_0}^{\infty} e^{-\kappa^2 \lambda} d\lambda \, \underline{I}^2 \underline{f}^\circ \underline{f}^\circ \tag{7.73}$$

which represents a transient behavior. The decreasing nature of

$$\partial \underline{f}^2 / \partial \tau_0$$

is immediately evident from the decreasing range of integration and we have in fact

$$\frac{\partial \underline{f}^2}{\partial \tau_0} \longrightarrow 0 \qquad \text{as} \quad \tau_0 \longrightarrow \infty \tag{7.74}$$

More precisely,

$$\frac{\partial \underline{f}^2}{\partial \tau_0} = O\left(\frac{1}{\tau_0^2}\right) \qquad \text{large } \tau_0 \tag{7.75}$$

The result (7.75) actually gives the asymptotic approach to the kinetic regime and we shall therefore derive it in detail.

We have, from (7.73),

$$\frac{\partial \underline{f}^2}{\partial \tau_0} = -2 \iint \underline{\nabla}_{12} U_{12} \underline{\nabla}_{V_1} \int_{\tau_0}^{\infty} e^{-\kappa^2 \lambda} d\lambda \, \underline{\nabla}_{12} U_{12} \left[\underline{\nabla}_{V_{12}} \underline{f}^\circ \underline{f}^\circ \right] d\underline{x}_1 \, d\underline{V}_{12} \tag{7.76}$$

where instead of the particle variables $\underline{V}_1$, and $\underline{V}_2$ we have

used $\underline{V}_1$, and $\underline{V}_{12}$ and $\underline{V}_1 - \underline{V}_2$. Therefore,

$$\frac{\partial f^2}{\partial \tau_0} = -2\nabla_{V_1}\int d\underline{V}_{12}\left[\nabla_{V_{12}} f^0 f^0\right]\int_{\tau_0}^{\infty} d\lambda \int \nabla_{12} U \nabla_{12} U(\underline{X}_{12} - \underline{V}_{12}\lambda)\, d\underline{X}_{12} \qquad (7.77)$$

but the tensor

$$T(\underline{V}_{12}\lambda) \equiv \int d\underline{X}_{12}\, \nabla_{12} U_{12} \nabla_{12} U(\underline{X}_{12} - \underline{V}_{12}\lambda)$$

$$= 0 \qquad \text{for} \quad |\underline{V}_{12}|\lambda \equiv v\lambda > 2r_0 \qquad (7.78)$$

by virtue of the overlap of two short range forces occurring in the integrand. Further

$$\int_{\tau_0}^{\infty} d\lambda\, T(\underline{V}_{12}\lambda) = \frac{1}{v}\int_{v\tau_0}^{\infty} d\underline{z}\, T(\underline{z}) \equiv \frac{1}{v}\tilde{T}(\underline{V}_{12}\tau_0) \qquad (7.79)$$

$$= 0 \qquad \text{for } v\tau_0 > 2r_0$$

We can therefore write

$$\frac{\partial f^2}{\partial \tau_0} = -2\nabla_{V_1} f^0(\underline{V}_1)\nabla_{V_1}\int d\underline{V}_{12} f^0(\underline{V}_1 - \underline{V}_{12}) : \frac{\tilde{T}(\underline{V}_{12}\tau_0)}{v} \qquad (7.80)$$

$$v < 2r_0/\tau_0$$

Thus, the integration over the relative velocities is confined to a shrinking sphere of radius $2r_0/\tau_0$. The formula (7.80) is exact. For large τ_0 we can replace it by

$$\frac{\partial f^2}{\partial \tau_0} \sim -2 \nabla_{V_1} \underline{f}^{\circ}(\underline{V}_1) \nabla_{V_1} \underline{f}^{\circ}(\underline{V}_1) \frac{1}{\tau_0^2} : \int_{w<2r_0} \frac{d\underline{w}}{w} \overline{T}(\underline{w}) \tag{7.81}$$

where we have used the mean value

$$\underline{f}^{\circ}(\underline{V}_1 - \underline{V}_2) = \underline{f}^{\circ}(\underline{V}_1) - \underline{V}_{12} \cdot \left[\frac{\partial \underline{f}^{\circ}(\underline{\xi})}{\partial \underline{\xi}}\right]_{\underline{\xi} \approx \underline{V}_1} \tag{7.82}$$

whose existence is required by the Liouville equation, and have observed that the term containing

$$\left[\partial \underline{f}^{\circ}(\underline{\xi})/\partial \underline{\xi}\right]_{\underline{\xi} \approx \underline{V}_1}$$

vanishes identically upon integration over $\underline{V}_{12}$.

To complete our description of the phase mixing mechanism that damps f^2 on the time scale τ_0, we observe that the rate of change of the entropy function

$$H = \int \underline{f} \log \underline{f} \, d\underline{V} \tag{7.83}$$

is given, correct to order ϵ^2, by

$$\frac{dH}{dt} - \left(\frac{dH}{dt}\right)_{asympt} = \epsilon^2 \iint \frac{d\underline{v}_1\, d\underline{v}_2}{\underline{f}_1^\circ\, \underline{f}_2^\circ} \int d\underline{x}_{12} \left(\underline{f}_2^\circ \nabla_{v_1} \underline{f}_1^\circ - \underline{f}_1^\circ \nabla_{v_2} \underline{f}_2^\circ \right) \cdot$$

$$\cdot \nabla_{12} U_{12} \left[\int_{\tau_0}^{\infty} e^{-\kappa^2 \lambda} d\lambda \right] \nabla_{12} U_{12} \left(\underline{f}_2^\circ \nabla_{v_1} \underline{f}_1^\circ \nabla_{v_2} f_2^\circ \right) \tag{7.84}$$

whose monotonic decrease is apparent from the positive definite nature of the integrand.

Noting that

$$\frac{\partial \underline{F}^{21}}{\partial \tau_1} = 0 \tag{7.85}$$

we obtain from (7.39) for the two-body distribution

$$\frac{\partial \underline{F}^{22}}{\partial \tau_0} + \kappa^2 \underline{F}^{22} = I^2 \int_0^{\tau_0} e^{-\kappa^2 \lambda} d\lambda\, I^2 \underline{f}^\circ \underline{f}^\circ +$$

$$+ L_2 \int_0^{\tau_0} e^{-\kappa^2 \lambda} d\lambda\, I^3 \overset{3}{\Pi} \underline{f}^\circ - \frac{\partial \underline{F}^{20}}{\partial \tau_2}$$

$$\widetilde{\tau}_0\, I_{12} (5^* I)_{12} \underline{f}^\circ \underline{f}^\circ + L_2\, \zeta_{123}^*\, I^3 \overset{3}{\Pi} \underline{f}^\circ - \tag{7.86}$$

$$- \left[L_{13} (5^* I)_{13} + L_{23} (5^* I)_{23} \right] \overset{3}{\Pi} \underline{f}^\circ$$

where use of the Landau equation (7.70) has been made to rewrite $\partial E^{20}/\partial \tau_2$. The quantity $L_2 (S^* I)^3 \Pi^3 f^0$ contains six terms:

a. The two "diagonal" terms cancel the $\partial E^{20}/\partial \tau_2$ correction to the time derivative.

b. Two are identically zero by the homogeneity of the gas.

c. Two "off diagonal" terms are new.

We have in fact

$$E^{22} \tilde{\tau}_0 (S^* I)_{12} (S^* I)_{12} f^0 f^0 + S_{12}^* \left[L_{13} (S^* I)_{23} + L_{23} (S^* I_{13}) \right] \Pi^3 f^0 \qquad (7.87)$$

For the three-body distribution we find from (7.43), after a calculation entirely analogous to the one just performed

$$E^{32} \tilde{\tau}_0 (S^* I)^3 (S^* I)^3 \Pi^3 f^0 +$$

$$+ S^{*3} \left[L_{14} (S^* I)_{24} + L_{14} (S^* I)_{34} + L_{24} (S^* I)_{14} + \right. \qquad (7.88)$$

$$\left. + L_{24} (S^* I)_{34} + L_{34} (S^* I)_{14} + L_{34} (S^* I)_{24} \right] \Pi^4 f^0$$

This completes the second order theory. One can further derive

$$E^{s2} \tilde{\tau}_0 (S^* I)^s (S^* I)^s \Pi^s f^0 +$$

$$+ S^{*s} \sum_{i=1}^{s} L_{i\,s+1} \sum_{\substack{j=1 \\ j \neq i}}^{s} (S^* I)_{j\,s+1} \Pi^{s+1} f^0$$

(v) Third Order Theory

From (7.35) and (7.87) we have

$$\frac{\partial \underline{f}^3}{\partial \tau_0} + \frac{\partial \underline{f}^2}{\partial \tau_1} + \frac{\partial \underline{f}^0}{\partial \tau_3}\,\tilde{\tau}_0 \tag{7.89}$$

$$\tilde{\tau}_0 L_{12}\{(\mathfrak{I}^* I)_{12}(\mathfrak{I}^* I)_{12}\underline{f}^0\underline{f}^0 + \zeta_{12}^*\left[L_{13}(\mathfrak{I}^* I)_{23} + L_{23}(\mathfrak{I}^* I)_{13}\right]\overset{3}{\Pi}\underline{f}^0\}$$

But, from (7.69) we have

$$\frac{\partial \underline{f}^2}{\partial \tau_1} = 0 \tag{7.90}$$

Therefore from (7.89) if $\epsilon^3\underline{f}^3$ is to remain a "small correction":

$$\frac{\partial \underline{f}^3}{\partial \tau_0}\,\tilde{\tau}_0 \sim 0 \tag{7.91}$$

and for the kinetic condition in third order:

$$\frac{\partial \underline{f}^0}{\partial \tau_3} = L_{12}(\mathfrak{I}^* I)_{12}(\mathfrak{I}^* I)_{12}\underline{f}^0\underline{f}^0 +$$

$$+ L_{12}\zeta_{12}^*\{L_{13}(\mathfrak{I}^* I)_{23} + L_{23}(\mathfrak{I}^* I)_{13}\}\overset{3}{\Pi}\underline{f}^0 \tag{7.92}$$

In (7.92), one sees two separate effects, the improved description of the binary collisions and the lowest order contribution from three-body collisions. We have in fact, schematically, for the collision effects:

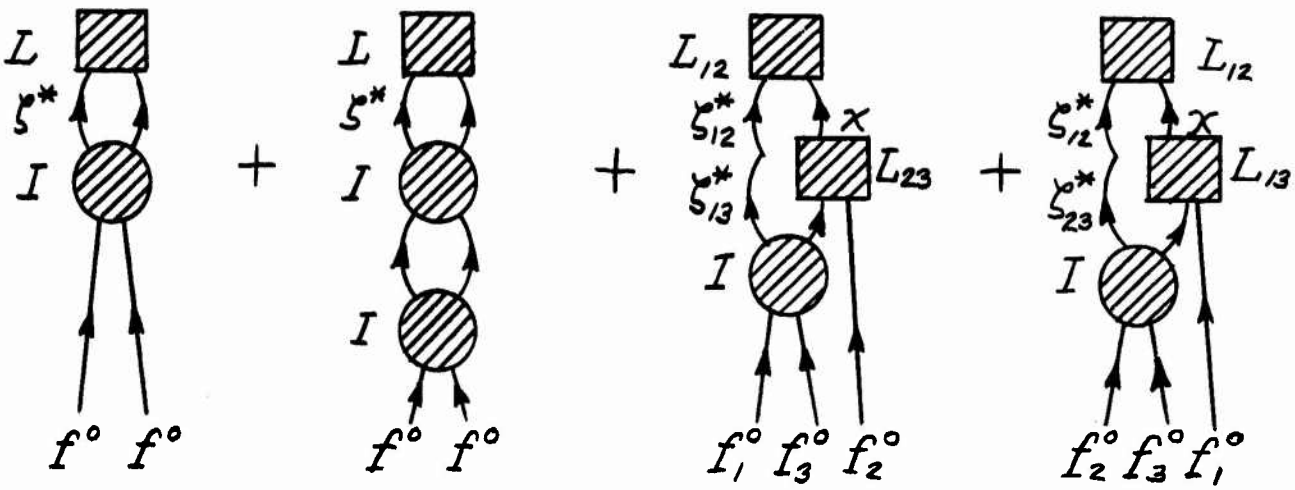

Fig. 12.

For the two-body distribution, (7.40) gives the third order equation:

$$\frac{\partial \underline{E}^{23}}{\partial \tau_0} + K^2 \underline{F}^{23} = I^2 \underline{F}^{22} + L_2 F^{32} - \frac{\partial \underline{F}^{21}}{\partial \tau_2} - \frac{\partial \underline{E}^{20}}{\partial \tau_3} \qquad (7.93)$$

We can therefore write:

$$\underline{E}^{23} \underset{\tau_0}{\sim} (5^* I)_{12} \underline{E}^{22\ as} + 5^*_{12} L_2 \underline{F}^{32\ as} -$$

$$- 5^*_{12} (5^* I)_{12} \frac{\partial \underline{f}^0 \underline{f}^0}{\partial \tau_2} - 5^*_{12} \frac{\partial \underline{f}^0 \underline{f}^0}{\partial \tau_3} \qquad (7.94)$$

where the superscript "as" indicates the asymptotic value of the function for large τ_0 . We have therefore for the successive contributions in (7.94), using (7.87), (7.88) of the second order theory

$$(5^* I)_{12} \underline{F}^{22,\,as} = (5^* I)_{12} (5^* I)_{12} (5^* I)_{12} \underline{f}^0 \underline{f}^0 +$$

$$+ (5^* I)_{12} 5^*_{12} \left[L_{13} (5^* I)_{23} + L_{23} (5^* I)_{13} \right] \overset{3}{\pi} \underline{f}^0 \qquad (7.95)$$

This contribution is <u>not</u> singular. Also

$$5^*_{12} (5^* I)_{12} \frac{\partial \underline{f}^0 \underline{f}^0}{\partial \tau_2} = 5^*_{12} (5^* I)_{12} \left[L_{13} (5^* I)_{23} + L_{23} (5^* I)_{13} \right] \overset{3}{\pi} \underline{f}^0 \qquad (7.96)$$

This contribution is very singular through the coinciding

singularities of the S^* functions. Furthermore:

$$S_{12}^* \frac{\partial \underline{f}^\circ \underline{f}^\circ}{\partial x_3} = S_{12}^* \left[L_{13} (S^* I)_{13} (S^* I)_{13} + L_{23} (S^* I)_{23} (S^* I)_{23} \right] \pi^3 \underline{f}^\circ +$$

$$+ S_{12}^* \left\{ L_{13} S_{13}^* \left[L_{14} (S^* I)_{34} + L_{34} (S^* I)_{14} \right] + L_{23} S_{23}^* \left[L_{24} (S^* I)_{34} + L_{34} (S^* I)_{24} \right] \right\} \pi^4 \underline{f}^\circ \qquad (7.97)$$

The trilinear terms are singular.

We finally consider the remaining contribution which, if the expansion were valid would cancel the singularities in (7.96) and (7.97)

$$S_{12}^* L_2 \underline{F}^{32\,as} = S_{12}^* \left\{ L_{13} (S^* I)_{123} (S^* I)_{123} + L_{23} (S^* I)_{123} (S^* I)_{123} \right\} \pi^3 \underline{f}^\circ +$$

$$+ S_{12}^* \left\{ L_2 S_{123}^* \left[L_{14} (S^* I)_{24} + L_{14} (S^* I)_{34} + L_{24} (S^* I)_{14} + \right. \right. \qquad (7.98)$$

$$\left. \left. + L_{24} (S^* I)_{34} + L_{34} (S^* I)_{14} + L_{34} (S^* I)_{24} \right] \right\} \pi^4 \underline{f}^\circ$$

We now consider the first contribution to the right hand side of (7.98) (of course our major interest is in the trilinear terms):

$$
\begin{aligned}
&5^{*}_{12} L_{13} (5^{*}I)_{123} (5^{*}I)_{123} \pi^{3} \underline{f}^{0} = \\
&= 5^{*}_{12} L_{13} 5^{*3} I_{12} \left[(5^{*}I)_{12} + (5^{*}I)_{13} + (5^{*}I)_{23} \right] \pi^{3} \underline{f}^{0} + \\
&+ 5^{*}_{12} L_{13} 5^{*3} I_{13} \left[(5^{*}I)_{12} + (5^{*}I)_{13} + (5^{*}I)_{23} \right] \pi^{3} \underline{f}^{0} + \\
&+ 5^{*}_{12} L_{13} 5^{*3} I_{23} \left[(5^{*}I)_{12} + (5^{*}I)_{13} + (5^{*}I)_{23} \right] \pi^{3} \underline{f}^{0}
\end{aligned}
\tag{7.99}
$$

Consider the singular contributions. The term

$$5^{*}_{12} L_{13} 5^{*3} I_{13} (5^{*}I)_{13} \pi^{3} \underline{f}^{0}$$

cancels one of the two singular contributions in (7.97). The other singular term in (7.97) is cancelled by the analogous term in the trilinear contribution to the right hand side of (7.98). Finally we consider the term

$$5^{*}_{12} L_{13} 5^{*3} I_{12} (5^{*}I)_{13} \pi^{3} \underline{f}^{0}$$

This term should cancel one of the two very singular contributions in (7.96). The cancellation fails however because the three-body propagator 5^{*3} does not reduce to a two-body propagator. The significance of this result will be clearer in the sequel when the short-range theory is discussed (Section 11) and the method of closure is defined.

(vi) Fourth-Order Theory

For the one-body distribution we find, using (7.64)

$$\frac{\partial \underline{f}^{4}}{\partial \tau_{0}} + \frac{\partial \underline{f}^{3}}{\partial \tau_{1}} + \frac{\partial \underline{f}^{2}}{\partial \tau_{2}} + \frac{\partial \underline{f}^{0}}{\partial \tau_{4}} = \angle \underline{F}^{23} \tag{7.100}$$

whence, following the same line of argument as in the third-order theory we obtain for the fourth-order kinetic condition:

$$\frac{\partial f^\circ}{\partial \tau_4} = L(S^*I)_{12}(S^*I)_{12}(S^*I)_{12}\, f^\circ f^\circ +$$

$$+ L(S^*I)_{12}\, S^*_{12}\left[L_{13}(S^*I)_{23} + L_{23}(S^*I)_{13}\right]\pi^3 f^\circ +$$

$$+ L\, S^*_{12}\left[L_{13}(S^*I)_{123}(S^*I)_{123} + L_{23}(S^*I)_{123}(S^*I)_{123}\right]\pi^3 f^\circ +$$

$$+ L\, S^*_{12}\left\{L_2\, S^*_{123}\left[L_{14}(S^*I)_{24} + L_{14}(S^*I)_{34} + L_{24}(S^*I)_{14} + \right.\right.$$

$$\left.\left. + L_{24}(S^*I)_{34} + L_{34}(S^*I)_{14} + L_{34}(S^*I)_{24}\right]\right\}\pi^4 f^\circ$$

$$- L\, S^*_{12}(S^*I)_{12}\left[L_{13}(S^*I)_{13} + L_{23}(S^*I)_{23}\right]\pi^3 f^\circ - \qquad (7.101)$$

$$- L\, S^*_{12}\left[L_{13}(S^*I)_{13}(S^*I)_{13} + L_{23}(S^*I)_{23}(S^*I)_{23}\right]\pi^3 f^\circ -$$

$$- L\, S^*_{12}\left\{L_{13}\, S^*_{13}\left[L_{14}(S^*I)_{34} + L_{34}(S^*I)_{14}\right] + \right.$$

$$\left. + L_{23}\, S^*_{23}\left[L_{24}(S^*I)_{34} + L_{34}(S^*I)_{24}\right]\right\}\pi^4 f^\circ$$

the various contributions are best visualized by means of the following graphs:

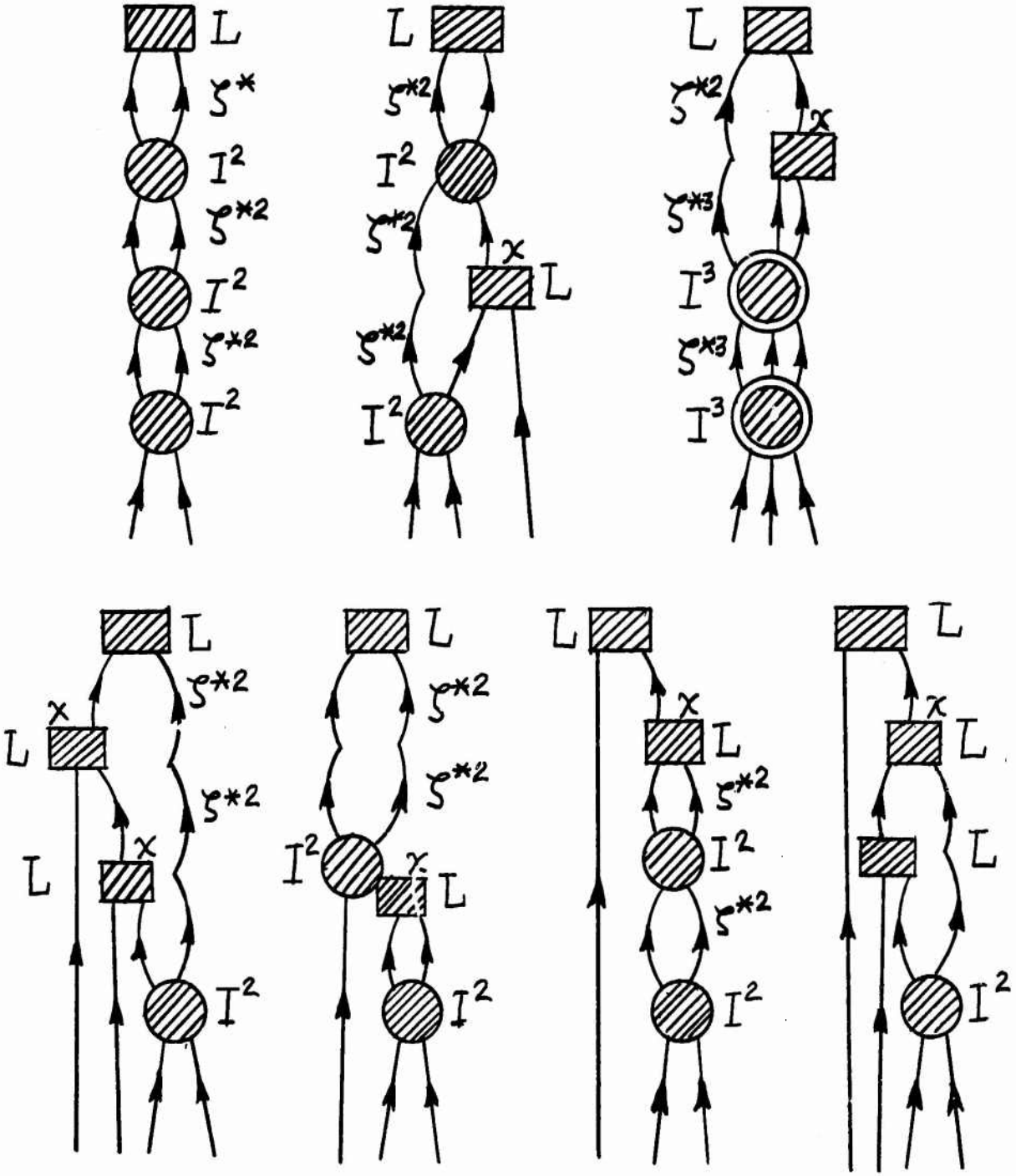

Fig. 13.

SECTION 8

THE PRINCIPLE OF ABSENCE OF PARALLEL MOTIONS

We now consider the complete initial value problem. The development of the expansion is very similar to that given for the "simple" initial value problem.

The initial conditions are assumed, however, to contain correlations which are arbitrary within the symmetry and normalization requirements imposed by the symmetry and normalization of the distribution functions. The correlations are introduced through a cluster expansion

$$F^1 = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + O(\varepsilon^3) \tag{8.1}$$

$$\begin{aligned} F^2 &= F^1 F^1 + g \\ &= (f^0 f^0 + g^0) + \varepsilon (f^0 f^1 + f^1 f^0 + g^1) + O(\varepsilon^2) \end{aligned} \tag{8.2}$$

$$F^3 = F^1 F^1 F^1 + \Sigma F^1 g + h \tag{8.3}$$

$$F^4 = \prod F^1 + \Sigma F^1 g F^1 + \Sigma F^1 h + k \tag{8.4}$$

The initial correlations correspond to disturbances imposed on the system of particles by means external to it. They are

therefore totally unrelated to the interparticle force as well as to the single particle velocity distribution. We accordingly expect them to have a finite, microscopic range R . The initial correlations will also have a characteristic velocity V_c, which for the purposes of our discussion need not bear any particular relation to V_{th}. We use the perturbation equation for the weak-coupling expansion (7.32) to (7.46).

(i) Zeroth Order Theory

We have

$$\frac{\partial f^0}{\partial \tau_0} = 0 \tag{8.5}$$

and

$$\underline{E}^{20}(\tau_0) = \underline{f}^0 \underline{f}^0 + e^{-K^2\tau_0} \underline{g}^0(0) \tag{8.6}$$

Similarly:

$$\underline{E}^{30}(\tau_0) = \overset{3}{\pi} \underline{f}^0 + \underline{f}_1^0 e^{-K_{23}\tau_0} \underline{g}_{23}^0(0) +$$

$$+ \underline{f}_2^0 e^{-K_{13}\tau_0} \underline{g}_{13}^0(0) + \underline{f}_3^0 e^{-K_{12}\tau_0} \underline{g}_{12}^0(0) + e^{-K^2\tau_0} \underline{h}^0(0) \tag{8.7}$$

(ii) First-Order Theory

The first-order single-particle distribution satisfies

$$\frac{\partial f'}{\partial \tau_0} + \frac{\partial f^0}{\partial \tau_1} = L e^{-K^2\tau_0} g^0(0) \tag{8.8}$$

In view of the finite range R of g^0 , we have

$$e^{-K^2\tau_0} g^0(0) \underset{\tau_0}{\sim} 0 \tag{8.9}$$

and

$$\frac{\partial \underline{f}^1}{\partial \tau_0} + \frac{\partial \underline{f}^0}{\partial \tau_1} \underset{\tau_0}{\sim} 0 \tag{8.10}$$

and since the integral of an asymptotic expansion is equal to the asymptotic expansion of the integral

$$\underline{f}^1(\tau_0) \underset{\tau_0}{\sim} \underline{f}^1(0) \sim \tau_0 \frac{\partial \underline{f}^0}{\partial \tau_1} \tag{8.11}$$

The non-uniformity is eliminated by the condition

$$\frac{\partial \underline{f}^0}{\partial \tau_1} = 0 \tag{8.12}$$

which implies that the rate of change of $\underline{f}^1$ is a transient on the τ_0 scale

$$\frac{\partial \underline{f}^1}{\partial \tau_0} = L e^{-K^2 \tau_0} \underline{g}^0(0) \underset{\tau_0}{\sim} 0 \tag{8.13}$$

We must, however, insure that F^1 remains normalizable as τ_0 increases, that is, we must insure that

$$\int_0^{\tau_0} L e^{-K^2 \lambda} \underline{g}^0(0)\, d\lambda$$

does not increase.

This normalization requirement implies that the initial two-body correlation does not allow for particle pairs with vanishing relative velocity. To obtain this basic result, we observe that we can apply the same overlap argument that we used to determine the asymptotic behavior of $\partial \underline{f}^2 / \partial \tau_0$ to the quantity

$$\begin{aligned} L e^{-K^2 \tau_0} \underline{g}^0(0) &= \iint \underline{\nabla}_1 U_{12} \underline{\nabla}_{V_1} e^{-K^2 \tau_0} \underline{g}^0(\underline{x}_{12}, \underline{V}_{12}, \underline{V}_1, 0)\, d\underline{x}_{12}\, d\underline{V}_{12} \\ &= \underline{\nabla}_{V_1} \int d\underline{V}_{12} \int d\underline{x}_{12}\, \underline{\nabla}_1 U_{12} \underline{g}^0(\underline{x}_{12} - \underline{V}_{12}\tau_0, \underline{V}_{12}, \underline{V}_1, 0) \end{aligned} \tag{8.14}$$

In fact,

$$J = \int \underline{V}_1 U_{12} f^0(\underline{x}_{12} - \underline{V}_{12}\tau_0, \underline{V}_{12}, \underline{V}_1, 0)\, d\underline{x}_{12}$$
$$= 0 \qquad \text{for } |\underline{V}_{12}|\tau_0 > r_0 + R \tag{8.15}$$

Therefore, the equation

$$L e^{-k^2\tau_0} f^0(0) = \underline{\nabla}_{V_1} \int_{r < \frac{r_0+R}{r_0}} d\underline{V}_{12}\, \underline{J} \tag{8.16}$$

emphasizes the importance of small relative velocities in the initial correlation for large τ_0. The condition that $\mathcal{E}'(\tau_0)$ should not increase affects the behavior of $f^0(\underline{x}_{12}, \underline{V}_{12}, \underline{V}_1, 0)$ in its relative velocity argument. We need, in fact, for any positive η

$$f^0(\underline{V}_{12}) \underset{|\underline{V}_{12}|\to 0}{\sim} \frac{|\underline{V}_{12}|^\eta}{V_{12}^2} \tag{8.17}$$

where the factor

$$\frac{1}{V_{12}^2} \equiv \frac{1}{|\underline{V}_{12}|^2}$$

represents the effect of the solid angle in relative velocities.

There are clearly two distinct classes of initial correlations; those for which $V_{12}^2 f^0$ vanishes for small relative velocities at least as strongly as V_{12}^η, the "kinetic" correlations, and those that fail to do so ("non-kinetic").

The "non-kinetic" correlations can be separated further into

two classes: "creeping" and "persistent". The correlations that violate (8.17) but nonetheless have a finite range cannot dominate the secular Landau collision term. A simple example of such a "creeping" initial correlation is offered by one that has a finite range and that behaves as

$$g^0 \sim \frac{1}{\sqrt{v_{12}}} \frac{1}{v_{12}^2} \tag{8.18}$$

for small relative velocities. This behavior is easily seen to be perfectly compatible with the symmetry and normalization conditions on g^0. For the corresponding $f^1(\tau_0)$ we have however the "creeping" behavior

$$f^1(\tau_0) \underset{\tau_0}{\sim} \sqrt{\tau_0} \tag{8.19}$$

Correlations which are either persistent or even dominate the kinetic behavior are those for which the quantity $K^2 g^0(0)$ is either zero or contains δ functions in iK^2. These correlations do not have a finite range and have a Fourier transform which contains δ functions in $\Delta \cdot V_{12}$. It is worth noting that the dynamical evolution of the system leads inevitably to such correlations in the limit of infinite time. This phenomenon emphasizes that a finite translation in time cannot be applied to the limit solutions.

For the two body distribution we find from (7.38)

$$\frac{\partial F^{21}}{\partial \tau_0} + K^2 F^{21} = I^2 f f^0 + I^2 e^{-K^2\tau_0} g^0(0) - e^{-K^2\tau_0} \frac{\partial g^0(0)}{\partial \tau_1} +$$
$$+ L_2 \left[f_1^0 e^{-K_{23}\tau_0} g_{23}^0(0) + f_2^0 e^{-K_{13}\tau_0} g_{13}^0(0) \right] + \tag{8.20}$$
$$+ L_2 e^{-K^3\tau_0} h^0(0)$$

The derivatives of the initial correlations with respect to the slower time scales are set equal to zero. This choice corresponds to the ambiguity that was introduced in the problem by extending functions of a single time variable t to functions on the independent time scales τ_n .

(iii) Second-Order Theory

To establish the principle of the absence of parallel motions in the initial three-body correlation, we consider the second-order equation

$$\begin{aligned}\frac{\partial f^2}{\partial \tau_0}+\frac{\partial f^1}{\partial \tau_1}+\frac{\partial f^0}{\partial \tau_2} &= L e^{-\kappa^2\tau_0} g'(0) + \\ &+ L\,\frac{1-e^{-\kappa^2\tau_0}}{\kappa^2} I^2 \underline{f}^0\underline{f}^0 + \\ &+ L e^{-\kappa^2\tau_0}\int_0^{\tau_0} e^{\kappa^2\lambda} I^2 e^{-\kappa^2\lambda} g^0(0)\,d\lambda + \\ &+ L e^{-\kappa^2\tau_0}\int_0^{\tau_0} e^{\kappa^2\lambda}\left[\underline{f}_1^0 e^{-\kappa_{23}\lambda} g_{23}^0(0)+\underline{f}_2^0 e^{-\kappa_{13}\lambda} g_3^0(0)\right]d\lambda \\ &+ L e^{-\kappa^2\tau_0}\int_0^{\tau_0} e^{\kappa^2\lambda} L_2 e^{-\kappa^3\lambda}\underline{h}^0(0)\,d\lambda\end{aligned} \tag{8.21}$$

The first term represents the phase mixing of the initial value of g^1 which is identical to that of g^0 . The second term

contains the persistent effect

$$L\zeta^*(i\kappa^2)I^2\underline{f}^0\underline{f}^0 \tag{8.22}$$

while the distorsion of the $\underline{g}^0$ correlation behaves asymptotically as

$$Le^{-\kappa^2\tau_0}I^2\zeta^*(i\kappa^2)\underline{g}^0(0)$$

which is strongly damped by the overlap of L with

$$e^{-\kappa^2\tau_0}I^2$$

The important term

$$\begin{aligned} &Le^{-\kappa^2\tau_0}\int_0^{\tau_0}e^{+\kappa^2\lambda}L_2e^{-\kappa^3\lambda}\underline{h}^0(0)d\lambda \\ &\underset{\tau_0}{\sim}Le^{-\kappa^2\tau_0}L_2\zeta^*(i\kappa^3)\underline{h}^0(0) \end{aligned} \tag{8.23}$$

is rewritten as

$$\begin{aligned} &Le^{-\kappa_{12}\tau_0}\Big[L_{13}\zeta^{*3}\underline{h}^0(\underline{x}_{13},\underline{x}_{13}-\underline{x}_{12})+ \\ &+L_{23}\zeta^{*3}\underline{h}^0(\underline{x}_{12}-\underline{x}_{23},\underline{x}_{23})\Big] \end{aligned} \tag{8.24}$$

which shows how L overlaps successively with the two separate position variables of $\underline{h}^0$. The three-body correlation is thereby required to satisfy (8.17) in both of its relative velocity variables. An entirely similar argument applies to the initial four-body correlation.

The equation for the two-body distribution is:

$$\frac{\partial \underline{F}^{22}}{\partial \tau_0} + K^2 \underline{F}^{22} =$$

$$I^2 \Big\{ \underline{f}'(0)\,\underline{f}^\circ(0) + \underline{f}^\circ(0)\,\underline{f}'(0) + e^{-K^2\tau_0}\underline{g}'(0) + \int_0^{\tau_0} e^{-K^2\lambda}\,d\lambda\, I^2 \underline{f}^\circ \underline{f}^\circ +$$

$$+ e^{-K^2\tau_0}\int_0^{\tau_0} e^{K^2\lambda} I^2 e^{-K^2\lambda}\,\underline{g}^\circ(0)\,d\lambda - \tau_0 e^{-K^2\tau_0}\frac{\partial \underline{g}^\circ(0)}{\partial \tau_1} +$$

$$+ e^{-K^2\tau_0}\int_0^{\tau_0} e^{K^2\lambda} L_2 \Big[\underline{f}_1^\circ e^{-K_{23}\lambda}\,\underline{g}_{23}^\circ(0) + \underline{f}_2^\circ e^{-K_{13}^\circ\lambda}\,\underline{g}_{13}^\circ(0) \Big] d\lambda + \qquad (8.25)$$

$$+ e^{-K^2\tau_0}\int_0^{\tau_0} e^{+K^2\lambda} L_2 e^{-K^3\lambda}\,\underline{h}^\circ(0)\,d\lambda \Big\}$$

$$- e^{-K^2\tau_0}\frac{\partial \underline{g}'(0)}{\partial \tau_1} + e^{-K^2\tau_0}\int_0^{\tau_0} e^{K^2\lambda} I^2 e^{-K^2\lambda}\frac{\partial \underline{g}^\circ(0)}{\partial \tau_1}\,d\lambda -$$

$$- \tau_0 e^{-K^2\tau_0}\frac{\partial^2 \underline{g}^\circ(0)}{\partial \tau_1^2} + e^{-K^2\tau_0}\int_0^{\tau_0} e^{K^2\lambda} L_2 \Big[\underline{f}_1^\circ e^{-K_{23}\lambda}\frac{\partial \underline{g}_{23}^\circ(0)}{\partial \tau_1} + \underline{f}_2^\circ e^{-K_{13}\lambda}\frac{\partial \underline{g}_{13}^\circ(0)}{\partial \tau_1} \Big] d\lambda$$

$$+e^{-\kappa^2\tau_0}\int_0^{\tau_0} e^{\kappa^2\lambda} L_2 e^{-\kappa^3\lambda}\frac{\partial \underline{h}^\circ(0)}{\partial \tau_1}d\lambda - e^{-\kappa^2\tau_0}\frac{\partial \underline{g}^\circ(0)}{\partial \tau_2} -$$

$$-\left[L_{13}(\zeta^* I)_{13} + L_{23}(\zeta^* I)_{23}\right]\overset{3}{\Pi}\underline{f}^\circ +$$

$$+L_2\left\{e^{-\kappa^3\tau_0}\left[\Sigma\underline{f}^\circ\underline{g}^2 + \Sigma\underline{f}'\underline{g}^\circ + \underline{h}'\right] + \int_0^{\tau_0} e^{-\kappa^3\lambda}d\lambda\, I^3\overset{3}{\Pi}\underline{f}^\circ +\right.$$

$$+e^{-\kappa^3\tau_0}\int_0^{\tau_0}\kappa^3\lambda\, I^3\Sigma\underline{f}^\circ e^{-\kappa^2\lambda}\underline{g}^\circ(0)\,d\lambda +$$

$$+e^{-\kappa^3\tau_0}\int_0^{\tau_0}\kappa^3\lambda\, I^3 e^{-\kappa^3\lambda}d\lambda\,\underline{h}^\circ(0) - \qquad (8.25)$$

$$-e^{-\kappa^3\tau_0}\int_0^{\tau_0} e^{+\kappa^3\lambda}\left[\Sigma\underline{f}^\circ e^{-\kappa^2\lambda}\frac{\partial \underline{g}^\circ(0)}{\partial\tau_1} + e^{-\kappa^3\lambda}\frac{\partial \underline{h}^\circ(0)}{\partial\tau_1}\right]d\lambda +$$

$$\left.+e^{-\kappa^3\tau_0}\int_0^{\tau_0}\kappa^3\lambda\, L_3 e^{-\kappa\kappa\lambda}\left[\Sigma\underline{f}^\circ\underline{g}^\circ\underline{f}^\circ + \Sigma\underline{f}^\circ\underline{h}^\circ + \underline{k}^\circ\right]d\lambda\right\}$$

We note that

$$\underline{E}^{22} \underset{\tau_0}{\approx} (\zeta^* I)_{12} [\underline{f}' \underline{f}^0 + \underline{f} \underline{f}^{0\prime}] +$$

$$+ \zeta^*_{12} \left[I^2 (\zeta^* I)^2 \underline{f}^0 \underline{f}^0 + \left(L_{13} \zeta^*_{13} I_{13} + L_{23} \zeta^*_{23} I_{23} \right) \overset{3}{\pi} \underline{f}^0 \right] \tag{8.26}$$

(iv) Third-Order Theory

Using (8.26) we have for the one-body function the following kinetic condition:

$$\frac{\partial \underline{f}^0}{\partial \tau_3} = L(\zeta^* I)(\underline{f}' \underline{f}^0 + \underline{f}^0 \underline{f}') + L(\zeta^* I)(\zeta^* I) \underline{f}^0 \underline{f}^0 +$$

$$+ L \zeta^*_{12} \left[L_{13} (\zeta^* I)_{23} + L_{23} (\zeta^* I)_{13} \right] \overset{3}{\pi} \underline{f}^0 \tag{8.27}$$

This formula differs from the result of the simple initial value problem by the first term only which represents the effect of the Landau collision operator on the first-order deviation of F^1 from Maxwellian.

(v) Discussion of the Results

We have shown that a classical gas will approach equilibrium through a kinetic equation only if the correlation functions satisfy certain conditions ("the absence of parallel motions") in their dependence on the relative velocity variables. These conditions, which represent the behavior of the correlations for small relative velocities, are derived from the Liouville equation and are therefore to be regarded as fundamental to kinetic theory.

The physical significance of the condition of absence of parallel motions is that, if a gas is to have a kinetic regime, then there cannot be too many statistically dependent particles which are at rest relative to each other.

The approach to equilibrium was shown to be simplest when the system is initially perfectly chaotic--i.e., when no correlations

whatever exist among the particles of the gas at the initial time (we call this initial conditions the "simple" initial value problem). Under these conditions, the approach to equilibrium of the initial gas is completely characterized by the behavior of a typical one of its particles, that is, the Liouville equation for the distribution of N bodies can be replaced by an equation for the evolution of the distribution of a single body.

It is clear, however, that we need not require that all initial correlations vanish in order to have a kinetic equation. The correlations must however satisfy some conditions which insure a sufficient amount of chaos in the initial state of the gas. We have shown that these are conditions on the behavior of the correlation functions for small values of the relative velocities. It is intuitively clear that the approach to equilibrium will be hindered if a gas contains a large number of particles with nearly equal velocities (i.e., a large number of particles which are at rest relative to each other).

In fact, it was shown that the Liouville equation imposes precise conditions on the initial correlations if a gas is to have a kinetic regime. Furthermore, when the conditions are violated, a kinetic description of the gas is not possible. These conditions are represented by asymptotic formulae for the initial correlations as functions of the relative velocities for small relative velocities. These formulae constitute a principle, (the principle of "absence of parallel motions") necessary and sufficient for the existence of a kinetic regime, that replaces the assumptions of molecular chaos and the Stosszahlansatz at the foundations of kinetic theory.

The mechanism by which a system of particles relaxes to equilibrium results, in the present theory, from our concentration on the behavior of the single body distribution function F^1, which satisfies the equation

$$\frac{\partial F^1}{\partial t} = L_1 F^N \tag{8.28}$$

as a consequence of the Liouville equation (1.3)

$$\frac{\partial F^N}{\partial t} + H^N F^N = 0 \tag{8.29}$$

for the distribution function of N bodies. The evolution of F^1 is time asymmetric as a consequence of choosing a particular initial distribution $F^N(0)$ whence F^1 has the rate of change

$$\frac{\partial F^1}{\partial t} = L_1 e^{-\mathcal{H}^N t} F^N(0) \tag{8.30}$$

This equation is time reflection invariant only for those $F^N(0)$ which have special symmetries in phase space.

The kinetic behavior of a gas is conditioned by the phase mixing that the operation of L_1 performs on the initial correlations through the collision mechanism represented by $\exp(-\mathcal{H}^N t)$. Only when "parallel motions" are absent in the initial correlations does the gas have a kinetic regime. In this case one has complete irreversibility--i.e., we are insured by the H-theorem that the system settles to thermodynamic equilibrium. The absence of parallel motions insures that the rate of change of F^1 with time will depend, after a short time, only on itself and not on the initial correlations present in the gas. $\partial F^1/\partial t$ is then "synchronized" to F^1, that is

$$L_1 e^{-\mathcal{H}^N t} F^N(0) \longrightarrow A[F^1] \qquad \text{only} \tag{8.31}$$

This is to be contrasted with the conditions for the synchronization of $F^s (s>1)$ to F^1. It is the vanishing of the initial correlations for large interparticle separations that insures the latter. This separate requirement of kinetic theory is expressed by

$$e^{-\mathcal{H}^N t} F^N(0) \longrightarrow B[F^1] \qquad \text{only} \qquad (8.32)$$

We see, therefore, that our condition establishes the role of the assumption made by Bogolubov in his fundamental investigation on the dynamical theory of gases.

The kinetic equations that we have derived are those obtained by Bogolubov. We have, however, gained insight into the manner in which a gas approaches the kinetic regime by actually deriving the conditions that make such a kinetic description of the system possible. We are therefore able to understand the circumstances that lead to the synchronization of F^2 and F^3 to F^1 on the one hand and for $\frac{\partial F^1}{\partial t}$ to F^1 on the other. The streaming boundary condition of Bogolubov is, in fact, a statement concerning the weakening of the correlations,

$$\overset{\circ}{S}{}^2_{-\infty} g = 0 \qquad (8.33)$$

which clearly insures the disappearance of all terms of both forms

$$e^{-\kappa^2 \tau_0} g, \quad L e^{-\kappa^2 \tau_0} g \qquad (8.34)$$

when a time large compared with r_0/v_{th} has elapsed.

From our formulae for the transient behavior of F^2 and F^3 on the τ_0 time scale, it is clear that the condition (8.33) insures the synchronization of the s-body distribution function to F^1 for large τ_0.

The Bogolubov synchronization cannot be valid in the entire

phase space. Thus, for example, in the simple initial value problem, when $\underline{V}_{12} = 0$

$$\underline{g}^1 = \underline{F}^{21} = \tau_0 I^2 \underline{f}^0 \underline{f}^0$$

i.e., a specific dependence on τ_0 persists in $\underline{F}^{21}$ and F^2 fails therefore to become synchronized to F^1 in some region of phase space.

The synchronization of $\frac{\partial F^1}{\partial t}$ to F^1 requires not only that the quantity

$$L e^{-K^2 \tau_0} g^0(0) \tag{8.35}$$

should vanish for large τ_0, as insured by (8.33), but further that it should disappear sufficiently fast to prevent its own time integral from growing. This can be insured only be giving $g^0(0)$ a <u>finite</u> range in the space variable and by requiring $\underline{g}^0(0)$ to satisfy the condition (8.34) (in the relative velocity variable). It is now clear why this condition, which we have derived from the Liouville equation and which requires the "absence of parallel motions" in the initial state of the system, is to be regarded as a fundamental requirement for kinetic theory.

APPENDIX

A SIMPLE EXAMPLE

We discuss a simple differential equation to indicate the full implications of extending a function beyond the needs of the expansion.

Consider the simple exponential

$$f(t) = e^{-\epsilon t} \tag{A1}$$

with $\epsilon < 1$. It satisfies

$$\dot{f} = -\epsilon f \tag{A2}$$

and

$$f(0) = 1 \tag{A3}$$

Clearly, the minimal extension:

$$f \Longrightarrow e^{-\tau_1} \tag{A4}$$

suffices the eliminate completely from the extended function the ϵ dependence. Thus, it would suffice to consider a two-dimensional extended domain (τ_0, τ_1) and the cylindrical extension as shown:

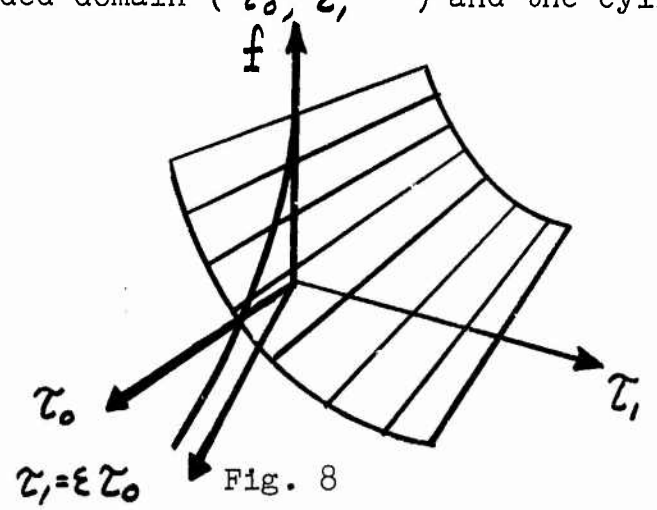

Fig. 8

Consider the Hilbert space

$$\tau_n = \epsilon^n t \tag{A5}$$

and the extensions

$$f^k(t) \Rightarrow \underline{f}^k(\underline{\tau}) \tag{A6}$$

where we set

$$f(t) = \sum_{k=0}^{\infty} \epsilon^k f^k(t) \tag{A7}$$

We find immediately

$$\frac{\partial \underline{f}^0}{\partial \tau_0} = 0 \tag{A8}$$

$$\frac{\partial \underline{f}^1}{\partial \tau_0} + \frac{\partial \underline{f}^0}{\partial \tau_1} = -\underline{f}^0 \tag{A9}$$

$$\frac{\partial \underline{f}^2}{\partial \tau_0} + \frac{\partial \underline{f}^1}{\partial \tau_1} + \frac{\partial \underline{f}^0}{\partial \tau_2} = -\underline{f}^1 \tag{A10}$$

and so on.

Let us confine ourselves for a moment to

$$\underline{f} \equiv \underline{f}^0 + \varepsilon \underline{f}^1 \tag{A11}$$

then we have

$$\underline{f}^{\circ} = e^{-\tau_1} \alpha(\tau_2, \tau_3, \ldots \tau_\infty) \tag{A12}$$

$$\underline{f}' = e^{-\tau_1} \left[f(\tau_2, \tau_3, \ldots, \tau_\infty) - \tau_1 \frac{\partial \alpha(\tau_2, \tau_3, \ldots \tau_\infty)}{\partial \tau_2} \right. \tag{A13}$$

and also

$$\frac{\partial \underline{f}'}{\partial \tau_n} + \frac{\partial \underline{f}^{\circ}}{\partial \tau_{n+1}} = 0 \qquad n \geq 2 \tag{A14}$$

Therefore

$$\frac{\partial \beta}{\partial \tau_2} - \tau_1 \frac{\partial^2 \alpha}{\partial \tau_2^2} + \frac{\partial \alpha}{\partial \tau_3} = 0 \tag{A15}$$

$$\frac{\partial \beta}{\partial \tau_3} - \tau_1 \frac{\partial^2 \alpha}{\partial \tau_2 \partial \tau_3} + \frac{\partial \alpha}{\partial \tau_4} = 0 \tag{A16}$$

$$\frac{\partial \beta}{\partial \tau_4} - \tau_1 \frac{\partial^2 \alpha}{\partial \tau_2 \partial \tau_4} + \frac{\partial \alpha}{\partial \tau_5} = 0 \tag{A17}$$

and so on. We must separate the linear term whence

$$\alpha(\tau_2, \ldots \tau_\infty) = A(\tau_3, \tau_4, \ldots \tau_\infty) + \tau_2 B(\tau_3 \ldots \tau_\infty) \tag{A18}$$

$$\frac{\partial B}{\partial \tau_n} = 0 \qquad n > 2 \tag{A19}$$

Therefore,

$$\alpha(\tau_2, \tau_3, \ldots, \tau_\infty) = A(\tau_3, \ldots \tau_\infty) + \tau_2 B^* \tag{A20}$$

where B^* is a true constant. The process can be continued indefinitely with the result that α and β are linear with well determined coefficients in all the $\tau_n (n > 2)$. Substituting in (A12) and (A13) then in (A11), one finds that these constants do not cancel, but restricting now with (A5) one simply has

$$f = e^{-\varepsilon t}(a + \epsilon b) \tag{A21}$$

(A3) then gives

$$a = 1, \; b = 0 \tag{A22}$$

If $\underline{f}^2$ is kept one has a much longer computation to do. The general answer is quadratic in the τ_n . But, after restriction, again the superfluous constants cancel.

If all the $\underline{f}^k$ are kept, one has to cancel not constants, but functions.

Closure can be invoked after the first approximation when there are no further non-uniformities in the expansion and one can therefore drop all the higher corrections to the time derivative.

REFERENCES, I

(1) M. J. Lighthill. Phil. Mag. 40, 1179, (1949).

(2) N. N. Bogolubov. Studies in statistical mechanics, vol. 1, edited by J. De Soer and G. E. Uhlenbeck, North Holland, (1962).

(3) S. T. Choh and G. E. Uhlenbeck. The kinetic theory of phenomena in dense gases. Navy Theor. Phys., NONR 1224 (15), (1958).

(4) This remark was made by J. R. Oppenheimer in the course of a discussion.

(5) Private communication to be published. Somewhat related ideas may be found in V. V. Nemytskii and V. Stepenov: Qualitative theory of differential equations, Princeton, 1960; M. Kruskal: Journal of Math. Phys. 3, 806 (62); R. Bellman and R. Kalaba: Proc. Ntl. Acad. Sci. U. S. 42 629 (56), 47 336 (61).

(6) G. Sandri and R. Sullivan. ARAP Working Paper 62-2, (1962).

(7) J. G. Kirkwood. J. Chem. Phys. 14, 180 (1946).

(8) A. Lenard. Ann. Phys. 10, 390 (1960).

(9) The wall potential is omitted here. It is discussed in references 2.

(10) Extensive references to the discoverers are in references 3.

(11) The idea of simplifying Bogolubov's theory for plasmas by making a time asymptotic analysis of the Liouville's equation is incorporated in M. Rosenbluth and N. Rostoker, Phys. Fluids, 3, 1, (1960). In this paper however an expansion valid for long times is not obtained.

(12) Many useful discussions with J. McCune, T. Morse, and R. Sullivan on the subject of this section are greatfully acknowledged.